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Cambridge International Advanced Level

FURTHER MATHEMATICS

9231/13

Paper 1

October/November 2016

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **15** printed pages.

Page 2	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more “method” steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
- The symbol ∇ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously “correct” answers or results obtained from incorrect working.
- Note: B2 or A2 means that the candidate can earn 2 or 0.
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

Page 3	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous’ answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become “follow through $\frac{1}{2}$ ” marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Page 4	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

1	$\frac{1}{(2r-1)(2r+1)} = \frac{1}{2} \left(\frac{1}{2r-1} - \frac{1}{2r+1} \right)$ $\sum_{r=1}^n \frac{1}{(2r)^2 - 1} = \frac{1}{2} \left(\left[1 - \frac{1}{3} \right] + \left[\frac{1}{3} - \frac{1}{5} \right] + \dots + \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right] \right) = \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \quad (\text{OE})$ $\frac{1}{2} \left(1 - \frac{1}{2n+1} \right) = \frac{n}{2n+1} \Rightarrow \sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} = \frac{1}{2}$	M1A1	[4]
		M1A1	
		B1	[1]
2	$2 \sum \alpha\beta = 9 - 1 \Rightarrow \sum \alpha\beta = 4$ Use of, e.g.: $\sum \alpha^3 - 3\alpha\beta\gamma = \sum \alpha (\sum \alpha^2 - \sum \alpha\beta)$ or $(\sum \alpha)^3 = \sum \alpha^3 + 3 \sum \alpha \sum \alpha\beta - 3\alpha\beta\gamma$ Correct substitution in formula $\Rightarrow \alpha\beta\gamma = -7$ Required cubic equation is $x^3 - 3x^2 + 4x + 7 = 0$ <i>must see final equation</i> ALT METHOD: $S_3 - 3S_2 + 4S_1 + 3r = 0$ M1 $3r = 30 + 3 \times 1 - 4 \times 3$ A1 $r=7$ A1	M1A1 M1 A1 A1 A1	[6]

Page 5	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

3	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ soi}$ $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} \Rightarrow \mathbf{A} = \mathbf{PDP}^{-1} \text{ soi}$ $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ $\mathbf{PD} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \text{ or } \mathbf{DP}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ $\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$ ALT METHOD: $\mathbf{Av}_1 = -v_1, \mathbf{Av}_2 = v_2, \mathbf{Av}_3 = 2v_3$ M1, A1 Multiply out, solve each equation M1, A1, A1, A1 $\mathbf{A} = \begin{pmatrix} -1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{B1} \checkmark$	B1B1 M1 M1A1 M1 A1	[7]
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Page 6	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

4	$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!} \left(\frac{1}{n-r+1} + \frac{1}{r} \right)$ $= \frac{n!}{(r-1)!(n-r)!} \left(\frac{r+n-r+1}{r(n-r+1)} \right) = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}$	M1	[2]
	$(a+x)^1 = \binom{1}{0}a + \binom{1}{1}x = a+x \Rightarrow H_1 \text{ is true.}$ Assume H_k is true, i.e. $(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \dots + \binom{k}{r}a^{k-r}x^r + \dots + \binom{k}{k}x^k$ Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$ $\Rightarrow H_{k+1}$ is true. Hence H_n is true for all positive integers.	B1 B1 M1 A1	

Page 7	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

5	(i)	$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 8 & 7 & 9 \\ 3 & 13 & 9 & 11 \\ 6 & 24 & 21 & 27 \end{pmatrix} \Rightarrow \dots \Rightarrow \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 2 & -3 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$ $r(\mathbf{A}) = 4 - 2 = 2$	M1A1 A1✓	[3]
	(ii)	Basis for range space is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ 13 \\ 24 \end{pmatrix} \right\}$ oe	B1	[1]
	(iii)	$x + 3y + 5z + 7t = 0$ $2y - 3z - 5t = 0$ Solving $\Rightarrow$ A basis for null space is $\left\{ \begin{pmatrix} -29 \\ 5 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -19 \\ 3 \\ 2 \\ 0 \end{pmatrix} \right\}$ oe	B1 M1 A1A1	[4]

Page 8	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

6	$m^2 + 7m + 10 = 0 \Rightarrow (m + 2)(m + 5) = 0 \Rightarrow m = -2 \text{ or } -5$ CF: $Ae^{-2t} + Be^{-5t}$ PI: $x = p \sin 2t + q \cos 2t \Rightarrow \dot{x} = 2p \cos 2t - 2q \sin 2t \Rightarrow x = -4p \sin 2t - 4q \cos 2t$ Substituting: $14p + 6q = 0$ $6p - 14q = 116$ Solving: $p = 3, q = -7 \Rightarrow x = 3 \sin 2t - 7 \cos 2t$ GS: $x = Ae^{-2t} + Be^{-5t} + 3 \sin 2t - 7 \cos 2t$ For large positive values of t $x \approx 3 \sin 2t - 7 \cos 2t$.	M1 A1 M1A1 M1 M1A1 A1 ^{1/2} B1 ^{1/2}	 [8] [1]

Page 9	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

7	(i)	$\text{M.V.} = \frac{\left[e^{-2x} \right]_0^2}{2-0} = \frac{e^{-4} - 1}{2} \quad (= -0.491)$	M1A1	[2]
	(ii)	Coordinates are $\bar{x} = \frac{\int_0^2 x e^{-2x} dx}{\int_0^2 e^{-2x} dx}$ and $\bar{y} = \frac{1}{2} \frac{\int_0^2 e^{-4x} dx}{\int_0^2 e^{-2x} dx}$ $\int_0^2 e^{-2x} dx = \left[-\frac{1}{2} e^{-2x} \right]_0^2 = \frac{1}{2} (1 - e^{-4}) \quad (= 0.4908\dots)$ (N.B. As in (i)) $\int_0^2 x e^{-2x} dx = \left[-\frac{x e^{-2x}}{2} \right]_0^2 + \int_0^2 \frac{e^{-2x}}{2} dx = \left[-\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} \right]_0^2 = -\frac{5e^{-4}}{4} + \frac{1}{4} (= 0.2271\dots)$ $\frac{1}{2} \int_0^2 e^{-4x} dx = \frac{1}{2} \left[-\frac{e^{-4x}}{4} \right]_0^2 = \frac{1}{2} \left(\frac{1}{4} - \frac{e^{-8}}{4} \right) \quad (= 0.12495\dots)$ (not straight from calculator) $\bar{x} = 0.463, \bar{y} = 0.255$ (Final answers in algebraic form SRA1)	M1M1 B1 M1A1 M1A1 A1A1	[9]

Page 10	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

8	$2x + 4(xy' + y) - 2yy' = 0 \quad (*)$ $y' = 0 \Rightarrow 2x + 4y = 0 \Rightarrow x = -2y \quad (AG)$ At stationary points $4y^2 - 8y^2 - y^2 + 20 = 0 \Rightarrow 5y^2 = 20 \Rightarrow y = \pm 2$ Coordinates of stationary points are (4, -2) and (-4, 2) From (*): $x + 2(xy' + y) - yy' = 0$ Differentiating: $1 + 2(xy'' + y' + y') - (yy'' + [y']^2) = 0$ (or by quotient rule) At (4, -2) with $y' = 0$: $1 + 8y'' + 2y'' = 0 \Rightarrow y'' = -\frac{1}{10} \Rightarrow$ maximum. At (-4, 2) with $y' = 0$: $1 - 8y'' - 2y'' = 0 \Rightarrow y'' = \frac{1}{10} \Rightarrow$ minimum.	M1A1 A1 M1A1 A1 M1A1 M1A1 A1	[3] [8]

Page 11	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

9	$\int_0^{\frac{\pi}{2}} x \sin x dx = [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx = 0 + [\sin x]_0^{\frac{\pi}{2}} = 1$	M1A1	[2]
	$I_n = [-x^n \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x dx$ $= [-x^n \cos x + nx^{n-1} \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x dx$ $= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$	M1A1 A1 A1	[4]
	$x = \cos^{-1} u \Rightarrow u = \cos x \Rightarrow \frac{du}{dx} = -\sin x ; u = 0, 1 \Rightarrow x = \pi/2, 0$ $\int_0^1 \left([\cos^{-1} u]^3 \right) du = - \int_{\pi/2}^0 x^3 \sin x dx = \int_0^{\pi/2} x^3 \sin x dx = I_3$ $I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3 \times 2 \times I_1 = \frac{3}{4}\pi^2 - 6 \text{ (OE)}$	B1B1 B1 M1A1	[5]

Page 12	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

10	$z^n = \cos n\theta + i \sin n\theta$ and $z^{-n} = \cos(-n\theta) + i \sin(-n\theta) = \cos n\theta - i \sin n\theta$ $\Rightarrow z^n + \frac{1}{z^n} = 2 \cos n\theta$ and $z^n - \frac{1}{z^n} = 2i \sin n\theta$. (AG)	M1A1	[2]
	$\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2 = \left(z - \frac{1}{z}\right)^2 \left(z^2 - \frac{1}{z^2}\right)^2$ (Or 1st bracket expanded) $= \left(z^2 - 2 + \frac{1}{z^2}\right) \left(z^4 - 2 + \frac{1}{z^4}\right)$ (and 2nd bracket expanded) $= \left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) + 4$ (Expanded and grouped) $\Rightarrow 64 \sin^4 \theta \cos^2 \theta = 2 \cos 6\theta - 4 \cos 4\theta - 2 \cos 2\theta + 4$ $\Rightarrow \sin^4 \theta \cos^2 \theta = \frac{1}{32} (\cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2)$ (AG) (Uses initial result)	M1 M1 M1A1 M1A1 A1	[7]
	$\int_0^{\frac{1}{4}\pi} \sin^4 \theta \cos^2 \theta \, d\theta = \frac{1}{32} \left[\frac{1}{6} \sin 6\theta - \frac{1}{2} \sin 4\theta - \frac{1}{2} \sin 2\theta + 2\theta \right]_0^{\frac{1}{4}\pi}$ $= \frac{1}{32} \left(-\frac{1}{6} - 0 - \frac{1}{2} + \frac{\pi}{2} \right) = \left(\frac{3\pi - 4}{192} \right) \text{ or } 0.0283 \text{ (3sf)}$ ALT METHOD: Expand and group each bracket separately and sub trig: M1, A1, A1 Multiply together two brackets and manipulate trig expressions M1, M1 LHS = RHS M1 A1	M1A1 A1	[3]

Page 13	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

11 E	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 3 & -4 & -2 \end{vmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ Obtains three equations. E.g. from $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ $t - 3s + 2k = 4$ $-3t + 4s - k = -2$ $-t + 2s + 5k = 4$ Solves: $s = -1$, $t = -1$, $k = 1$ $\mathbf{p} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ (Both required) ALT METHOD: Find PQ in terms of s,t M1 Calculate scalar products direction vectors for l_1 and l_2 M1,A1,A1 Solve simultaneous equations M1 A1 p and q A1 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -2 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix}$ $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -22 \\ -19 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$	M1A1	[7]
		M1A1 M1A1 A1 B1 B1B1	
			[3]

Page 14	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

	Direction of l_3 is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -2 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} -4 \\ -3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$	B1	[4]
	Normal to Π is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 22 & 19 & -5 \\ 2 & -1 & 5 \end{vmatrix} = \begin{pmatrix} 90 \\ -120 \\ -60 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$	M1	
	Cartesian equation of Π is $3x - 4y - 2z = 1$ (Since P lies in Π)	A1	
	Vector equation of l_3 is $\mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ (OE) (Since $z = 0$.)	B1	

Page 15	Mark Scheme	Syllabus	Paper
	Cambridge International A Level – October/November 2016	9231	13

11 O	$\dot{x}^2 + \dot{y}^2 = 36t^2 + 1 - 18t^2 + 81t^4 = 1 + 18t^2 + 81t^4 = (1 + 9t^2)^2 \quad (\text{AG})$	M1A1	[2]
(i)	$s = \int_0^{\frac{1}{\sqrt{3}}} (1 + 9t^2) dt = \left[t + 3t^3 \right]_0^{\frac{1}{\sqrt{3}}} = \frac{2}{\sqrt{3}}$	M1A1	[2]
(ii)	$S = 2\pi \int_0^{\frac{1}{\sqrt{3}}} t(1 - 3t^2)(1 + 9t^2) dt = 2\pi \int_0^{\frac{1}{\sqrt{3}}} (t + 6t^3 - 27t^5) dt$ $= 2\pi \left[\frac{1}{2}t^2 + \frac{3}{2}t^4 - \frac{9}{2}t^6 \right]_0^{\frac{1}{\sqrt{3}}} = \frac{\pi}{3}$ Substitute $t = \frac{y}{x}$ in one of the parametric equations to obtain e.g. $x = 1 - 3\frac{y^2}{x^2}$. Uses $x = r \cos \theta$ and $y = r \sin \theta \Rightarrow r \cos \theta = 1 - 3 \tan^2 \theta \Rightarrow r = \sec \theta (1 - 3 \tan^2 \theta)$. States domain for θ: $0 \leq \theta \leq \frac{\pi}{6}$, $A = \frac{1}{2} \int_0^{\frac{\pi}{6}} \sec^2 \theta (1 - 6 \tan^2 \theta + 9 \tan^4 \theta) d\theta$ $= \frac{1}{2} \left[\tan \theta - 2 \tan^3 \theta + \frac{9}{5} \tan^5 \theta \right]_0^{\frac{\pi}{6}} = \frac{4}{45} \sqrt{3}$	*M1 DM1A1 M1 M1A1 B1 M1 A1A1	[3] [4] [3]