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Paper 2

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MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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PUBLISHED**Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1:

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

GENERIC MARKING PRINCIPLE 2:

Marks awarded are always **whole marks** (not half marks, or other fractions).

GENERIC MARKING PRINCIPLE 3:

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

GENERIC MARKING PRINCIPLE 4:

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

GENERIC MARKING PRINCIPLE 5:

Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

GENERIC MARKING PRINCIPLE 6:

Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mark Scheme Notes

Marks are of the following three types:

- M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.
 - A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).
 - B Mark for a correct result or statement independent of method marks.
- When a part of a question has two or more ‘method’ steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
 - The symbol FT implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously ‘correct’ answers or results obtained from incorrect working.
 - Note: B2 or A2 means that the candidate can earn 2 or 0.
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

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The following abbreviations may be used in a mark scheme or used on the scripts:

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no ‘follow through’ from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous’ answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become ‘follow through’ marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Guidance
1	<i>EITHER:</i> $m(250 - 70) = 450 \times 0.04$	M1 A1	Find m from change in momentum = Ft
	$m = \frac{18}{180} = 0.1$	A1	(can allow M1 if $250 + 70$ used; similarly below)
	<i>OR:</i> $a = (250 - 70)/0.04 [= 4500]$	(M1)	Find deceleration a (ignore sign)
	$m = \frac{450}{a} = 0.1$	(M1 A1)	Find m from $F = ma$
		3	

Question	Answer	Marks	Guidance
2(i)	$v_A^2 = \omega^2(a^2 - 1.6^2)$ and $v_B^2 = \omega^2(a^2 - 1.2^2)$	B1	Use $v^2 = \omega^2(a^2 - x^2)$ at A and B (may be implied)
	$\frac{9}{16} = \frac{(a^2 - 1.6^2)}{(a^2 - 1.2^2)}$ or $\frac{(a^2 - 2.56)}{(a^2 - 1.44)}$	M1 A1	Find eqn. for a from ratio of v_A to v_B
	$7a^2 = 40.96 - 12.96$ or 10×2.8 , $a^2 = 4$, $a = 2$ [m]	A1	and hence a (error in $\frac{9}{16}$ will lose all A1s, so max $\frac{4}{11}$)
		4	
2(ii)	$\omega = \frac{1}{3} \frac{\pi}{a} = \frac{\pi}{6}$, $T = \frac{2\pi}{\omega} = 12$ [s]	M1 A1	Find ω and hence period T from $v_{max} = \omega a$ and $T = \frac{2\pi}{\omega}$
		2	

Question	Answer	Marks	Guidance
2(iii)	$\omega^{-1} \sin^{-1}\left(\frac{1.6}{a}\right) + \omega^{-1} \sin^{-1}\left(\frac{1.2}{a}\right) [= \omega^{-1} \frac{\pi}{2}]$ <i>or</i> $\omega^{-1} \cos^{-1}\left(\frac{-1.6}{a}\right) - \omega^{-1} \cos^{-1}\left(\frac{1.2}{a}\right)$ <i>or</i> $\frac{1}{2}T - \omega^{-1} \cos^{-1}\left(\frac{1.6}{a}\right) - \omega^{-1} \cos^{-1}\left(\frac{1.2}{a}\right)$	M1	Find time from A to B from $x = a \sin \omega t$ or $a \cos \omega t$ All terms must be correct (FT on a , ω) for M1
	$= \omega^{-1} (0.9273 + 0.6435)$ <i>or</i> $\omega^{-1} (2.498 - 0.9273)$ <i>or</i> $6 - \omega^{-1} (0.6435 + 0.927)$ (AEF throughout)	A1	
	$= 1.771 + 1.229$ <i>or</i> $4.771 - 1.771$ <i>or</i> $6 - 1.229 - 1.771,$	A1	(OR use geometry of circular motion with radius a and angular velocity ω)
		3	

Question	Answer	Marks	Guidance
3(i)	$mv_A = mu - mku$ (AEF)	M1	Use conservation of momentum (m may be omitted)
	$v_A = -e(u + ku)$	M1	Use Newton's restitution law
	$-e(1 + k) = 1 - k, e = \frac{(k-1)}{(k+1)}$ AG	A1	Combine to find/verify e
		3	

Question	Answer	Marks	Guidance
3(ii)	$KE_I - KE_F = 0.6 KE_I \text{ or } 0.4 KE_I = KE_F$	M1	Attempt to relate initial and final KEs
	$0.4 \times \frac{1}{2} mu^2 (1 + k^2) = \frac{1}{2} mv_A^2 = \frac{1}{2} mu^2 (1 - k)^2$	M1	Substitute for KEs ($\frac{1}{2} m$ may be omitted)
	$0.6k^2 - 2k + 0.6 = 0 \text{ or } 3k^2 - 10k + 3 = 0$ (AEF)	A1	Simplify to quadratic eqn in k
	$(3k - 1)(k - 3) = 0$ so solutions are 3 and $\frac{1}{3}$	A1	Find both solutions of quadratic eqn
	[Reject $k = \frac{1}{3}$] so $k = 3$	A1	(Implicitly) reject one solution to find possible value of k since $0 < e < 1$ or $v_A < 0$
	$e = \frac{2}{4} = \frac{1}{2}$ (FT on k)	A1√	Find single corresponding value of e
		6	

Question	Answer	Marks	Guidance
4(i)	$R_B 2a \sin \theta = Wa \cos \theta + 5Wxa \cos \theta$	M1 A1	Moments at A
	$R_B = W \frac{(5x+1)}{2 \tan \theta}$ AG	A1	Simplify to give R_B
		3	
4(ii)	$R_B' = W \frac{(5x+5+1)}{2 \tan \theta}$ (AEF)	B1	Find new R_B' by moments at A
	$5x + 6 = 2(5x + 1), x = \frac{4}{5}$ AG	M1 A1	Find/verify x from $R_B' = 2 R_B$
		3	

Question	Answer	Marks	Guidance
4(iii)	$R_A = 6W, F_A' = R_B' \text{ or } 2 R_B [= \frac{10W}{2 \tan \theta} = 4W]$	B1, B1	Find R_A and new F_A' by resolutions (may be earned earlier)
	$\frac{2}{3} = \frac{F_A'}{R_A}$	M1	Relate R_A and F_A' using $\mu = \frac{F}{R}$
	$= \frac{\left(\frac{5W}{\tan \theta} \right)}{6W} \text{ so } \tan \theta = \frac{5}{4} \text{ or } 1.25$	M1 A1	Find $\tan \theta$ SC Allow M1 M1 for ground smooth and wall rough ($\max \frac{2}{5}$)
		5	

Question	Answer	Marks	Guidance
5(i)	$I_A = \frac{1}{2} 3M (2a)^2$ [= 6 Ma^2]	*M1	Find MI of A // to axis l at A 's centre by $\perp$ axis theorem
	$I_A' = I_A + 3M (2a)^2$ [= 18 Ma^2] (dep *M1)	M1 A1	Find MI of A about axis l
	$I_B = \frac{1}{2} 2M (3a)^2$ [= 9 Ma^2]	**M1	Find MI of B (or C) // to axis l at its centre by $\perp$ axis theorem
	$I_B' = I_B + 2M (6a)^2$ [= 81 Ma^2] (dep **M1)	M1 A1	Find MI of B (or C) about axis l
	$I = (18 + 2 \times 81) Ma^2 = 180 Ma^2$ AG	A1	Verify MI of object about axis l (A0 if inadequate explanation)
	SC: $I_A' = \frac{1}{2} \{3M (2a)^2 + 3M (2a)^2\}$ [= 12 Ma^2]	(M1)	SC: Invalidly applying theorems in wrong order
	$I_B' = \frac{1}{2} \{2M (3a)^2 + 2M (6a)^2\}$ [= 45 Ma^2]	(M1)	(max $\frac{2}{7}$)
		7	

Question	Answer	Marks	Guidance
5(ii)	$[-] I \frac{d^2\theta}{dt^2} = 3Mg \times 2a \sin \theta + 2 \times 2Mg \times 6a \sin \theta$ $\text{or } 7Mg \times \frac{30a}{7} \sin \theta \quad [= 30Mga \sin \theta]$	*M1 A1	Use eqn of circular motion to find $d^2\theta/dt^2$ where θ is angle of plane of object with vertical
	$\frac{d^2\theta}{dt^2} = -\left(\frac{g}{6a}\right)\theta \text{ or } -\left(\frac{0.167g}{a}\right)\theta \quad (\text{M1 dep *M1})$	M1 A1	Approximate $\sin \theta$ by θ to show SHM (no ‘–’ is M1 A0)
	$T = \frac{2\pi}{\sqrt{\left(\frac{g}{6a}\right)}}$ $= 2\pi\sqrt{\left(\frac{6a}{g}\right)} \text{ or } 15.4\sqrt{\left(\frac{a}{g}\right)} \text{ or } 4.87\sqrt{a} \quad (\text{AEF})$	B1✓	Find period T from $T = 2\pi/\omega$ (FT on ω^2 ; requires some simplification)
		5	

Question	Answer	Marks	Guidance
6(i)	$P(X > 2) = 1 - F(2) = \exp(-0.8) = 0.449$	M1 A1	Find $P(X > 2)$. M0 for $F(2)$
		2	
6(ii)	$F(Q) = 1 - \exp(-0.4x) = \frac{1}{4} \text{ or } \frac{3}{4}$	M1	Formulate equation for either quartile value Q
	$Q_1 = (\ln 4/3) / (0.4) \quad [= 0.7192] \quad (\text{AEF})$	A1	Find one [lower] quartile Q_1
	$Q_3 = (\ln 4) / (0.4) \quad [= 3.466] \quad (\text{AEF})$	A1	Find other [upper] quartile Q_3
	$Q_3 - Q_1 [= (\ln 3) / (0.4)] = 2.75$	A1 √	Find interquartile range (FT on Q_1, Q_3 ; allow $Q_1 - Q_3$)
		4	

Question	Answer	Marks	Guidance
7	$\bar{x} = 681.6/7 = 97.37 \quad (\text{allow } 97.4 \text{ for this B1})$	B1	Find sample mean
	$s^2 = (66\,536.1 - 681.6^2/7) / 6$ [= 27.96 or 5.287 ²]	M1	Estimate population variance (allow biased here: 23.96 or 4.895 ²)
	$H_0: \mu = 94, H_1: \mu \neq 94 \quad (\text{AEF})$	B1	State hypotheses (B0 for $\bar{x} \dots$)
	$t = (\bar{x} - 94)/(s/\sqrt{7}) = 1.69$	M1 A1	Find value of t
	$t_{7, 0.95} = 1.94[3]$	B1	State or use correct tabular t -value (or can compare $\bar{x}$ with $94 + 3.88 = 97.88$)
	[Accept H_0 :] Mean mass is equal to 94 kg (AEF)	B1 √	Consistent conclusion (FT on both t -values)
		7	

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Question	Answer	Marks	Guidance
8	H_0 : Colour is independent of type or no association between colour and type (AEF)	B1	State (at least) null hypothesis in full
	E_i : $\begin{matrix} 47.67 & 43.33 & 39 \\ 36.67 & 33.33 & 30 \\ 25.67 & 23.33 & 21 \end{matrix}$ (to 3 s.f.)	M1 A1	Find expected values E_i (A0 if rounded to integers)
	$X^2 = 0.597 + 1.241 + 0.103$ $+ 1.603 + 1.333 + 0.033$ $+ 0.212 + 0.019 + 0.429 = 5.57$ (to 3 s.f.)	M1 A1	Find value of χ^2 from $\Sigma (E_i - O_i)^2 / E_i$ [or $\Sigma O_i^2 / E_i - n$] (allow 5.64 for this A1 if integer values of E_i used)
	$\chi_{4, 0.9}^2 = 7.779$ or 7.78	B1	State or use correct tabular χ^2 value
	Accept H_0 if $X^2 < \text{tabular value}$ (AEF)	M1	Valid method for reaching conclusion
	$5.57 [\pm 0.01] < 7.78$ so independent or no association (AEF)	A1	Correct (abbreviated) conclusion, from approx. correct values
		8	

Question	Answer	Marks	Guidance
9(i)	$(1-p)/p^2 = 4/9, \quad 4p^2 + 9p - 9 = 0$ AG	M1 A1	Find given eqn. for p using $\text{Var}(X) = (1-p)/p^2$
	$(4p-3)(p+3) = 0, \quad p = 3/4$	M1 A1	Solve quadratic for p (A0 if $p = -3$ not [implicitly] rejected)
		4	
9(ii)	$P(X=3) = (1-p)^2 p = (1/4)^2 \cdot 3/4 = 3/64$ or 0.0469	B1	Find $P(X=3)$
		1	

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Question	Answer	Marks	Guidance
9(iii)	$P(X \geq 4) = (1 - p)^3 = (\frac{1}{4})^3$ or $1 - (1 + \frac{1}{4} + (\frac{1}{4})^2)^{\frac{3}{4}} = 1/64$ or 0.0156	M1 A1	Find $P(X \geq 4)$
		2	
9(iv)	$1 - (1 - p)^N > 0.999$	M1	Formulate condition for N ($1 - (1 - p)^{N-1}$ is M0)
	$0.001 > (\frac{1}{4})^N$	A1	(< or = can earn M1 M1 only, max 2/4)
	$N > \log 0.001 / \log 0.25$	M1	Rearrange and take logs (any base) to give bound
	$N > 4.98, N_{\min} = 5$	A1	Find $N_{\min}$
		4	

Question	Answer	Marks	Guidance
10(i)	$H_0: \mu_P = \mu_Q, H_1: \mu_P > \mu_Q$	B1	State hypotheses (B0 for $\bar{x} \dots$)
	$s_P^2 = (138\,200 - 2620^2/50) / 49 = 18.61$ or $912/49$ and $s_Q^2 = (157\,000 - 3060^2/60) / 59 = 15.93$ or $940/59$	M1 A1	Estimate both popn. variances (to 3 s.f.) (allow biased here: 18.24 and 15.67)
	$s^2 = s_P^2/50 + s_Q^2/60 = 0.6378$ or 0.7986^2	M1 A1	Estimate combined variance (to 3 s.f.; may be implicit)
	$z_{0.9} = 1.28[2]$	*B1	State or use correct tabular z value
	$z = (\bar{y} - \bar{x}) / s = (52.4 - 51) / s = 1.75$ $z > \text{tabular value so [accept } H_1 \text{ and]}$	M1 A1	Calculate value of z (or $-z$) (or can compare $\bar{y} - \bar{x} = 1.02$ with 1.4)
	college P students take more time (AEF)	B1✓	Correctly stated conclusion (FT on z , dep *B1)
	[SC: $s^2 = (912 + 940)/108 = 17.15$ or 4.141^2 $z = 1.4 / s\sqrt{(1/50 + 1/60)} = 1.77$]		SC: Using pooled estimate of common variance can earn B1 M1 A1 (may be implied) M0 *B1 M0 B1✓ (max 5/9)
		9	
10(ii)	$\bar{x} - \bar{y} \pm z s$ (or $\bar{y} - \bar{x} \pm z s$)	M1	Find confidence interval for difference
	$z_{0.95} = 1.64[5]$	A1	Use appropriate tabular value
	1.4 ± 1.31 or $[0.09, 2.71]$ or -1.4 ± 1.31 or $[-2.71, -0.09]$ [SC: e.g. $\bar{x} - \bar{y} \pm z s\sqrt{(1/50 + 1/60)} = 1.4 \pm 1.30$]	A1	Evaluate confidence interval (either form) SC: Allow M1 A1 if common variance used (max 2/3)
		3	

Question	Answer	Marks	Guidance
11A(i)	$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mga \cos \theta$ and	M1	Use conservation of energy (A0 if no m) and
	$T = mv^2/a + mg \cos \theta$ $T = mu^2/a + 3mg \cos \theta$	M1 A1	find tension T at P_1 by using $F = ma$ (A1 for both correct) Combine to verify $\cos \theta$
	$(11/5)mg = 2mg/5 + 3mg \cos \theta$, $\cos \theta = 3/5$ AG	M1 A1	by substituting $T = (11/5)mg$ and $u = \sqrt{(2ag/5)}$
	$v^2 = u^2 + 2ag \cos \theta = (2/5 + 6/5)ag = 8ag/5$ $v = \sqrt{(8ag/5)}$ or $\sqrt{(1.6ag)}$ or $4\sqrt{a}$ (AEF)	B1	Find v at P_1 (can assume $\cos \theta = 3/5$)
		6	
11A(ii)	$t = (a \sin \theta) / (v \cos \theta) = 4a/3v$ [= $\frac{1}{3}\sqrt{a}$]	M1 A1	Find time t to P_2 by considering horizontal motion
	$h = (v \sin \theta) t + \frac{1}{2}gt^2$ $= 16a/15 + 5a/9 = 73a/45$ or $1.62a$	M1 A1	Find ht. fallen at P_2 by considering vertical motion
	$OP_2 = h + a \cos \theta = 20a/9$ or $2.22a$	M1 A1	Find OP_2
		6	
11B(i)	$\Sigma x = 48$, $\Sigma y = 29 + 2p$, $\Sigma xy = 242 + 17p$, $\Sigma x^2 = 450$, $[\Sigma y^2 = 219 + 2p^2]$ $S_{xy} = 242 + 17p - 48 \times (29 + 2p)/6 = 10 + p$ $S_{xx} = 450 - 48^2/6 = 66$ $[S_{yy} = 219 + 2p^2 - (29 + 2p)^2/6 = (8p^2 - 116p + 473)/6]$	M1 A1	Find required values
	$0.25 = S_{xy} / S_{xx} = (10 + p)/66$, $p = 6.5$	M1 A1	Find p from gradient in eqn. of regression line
	$(29 + 2p)/6 = 0.25 \times 48/6 + k$, $k = 7 - 2 = 5$	M1 A1	Find k from means and regression line
		6	

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Question	Answer	Marks	Guidance
11B(ii)	$[\Sigma y = 42, \Sigma y^2 = 303.5, S_{yy} = 303.5 - 42^2/6 = 9.5]$ $r = S_{xy} / \sqrt{(S_{xx} S_{yy})} = 16.5 / \sqrt{(66 \times 9.5)} = 0.659$	M1 *A1	Find correlation coefficient r
		2	
11B(iii)	$H_0: \rho = 0, H_1: \rho > 0$	B1	State both hypotheses (B0 for $r \dots$)
	<i>EITHER:</i> $r_{6, 5\%} = 0.729$	*B1	State or use correct tabular one-tail r -value
	Accept H_0 if $ r < \text{tab. } r\text{-value}$ (AEF)	M1	State or imply valid method for conclusion
	<i>OR:</i> $t_r = r\sqrt{(n-2)/(1-r^2)} = 1.75, t_{4,0.95} = 2.132$	(*B1)	(Rarely seen)
	Accept H_0 if $ t_r < \text{tab. } t\text{-value}$ (AEF)	(M1)	
	No positive correlation (AEF)	A1	Correct conclusion (dep *A1, *B1)
		4	