

1.  $f(x) = x^4 - x^3 - 9x^2 + 29x - 60$

Given that  $x = 1 + 2i$  is a root of the equation  $f(x) = 0$ , use algebra to find the three other roots of the equation  $f(x) = 0$

(7)

$$x = 1 \pm 2i$$

$$x - 1 = \pm 2i$$

$$(x - 1)^2 = -4$$

$$x^2 - 2x + 5 = 0$$

$$\begin{array}{r}
 x^2 - 2x + 5 \quad \overline{) \quad x^4 - x^3 - 9x^2 + 29x - 60} \\
 \underline{-(x^4 - 2x^3 + 5x^2)} \\
 x^3 - 14x^2 + 29x \\
 \underline{-(x^3 - 2x^2 + 5x)} \\
 -12x^2 + 24x - 60 \\
 \underline{-(-12x^2 + 24x - 60)} \\
 0
 \end{array}$$

$$(x^2 - 2x + 5)(x^2 + x - 12) = x^4 - x^3 - 9x^2 + 29x - 60$$

$$x = 1 \pm 2i \quad \text{or} \quad x = 3 \quad \text{or} \quad -4$$

$$x = 3$$

$$x = -4$$

$$x = 1 - 2i$$

2.

$$f(x) = x^3 - 3x^2 + \frac{1}{2\sqrt{x^5}} + 2, \quad x > 0$$

(a) Show that the equation  $f(x) = 0$  has a root  $\alpha$  in the interval  $[2, 3]$ .

(2)

(b) Taking 3 as a first approximation to  $\alpha$ , apply the Newton-Raphson process once to  $f(x)$  to find a second approximation to  $\alpha$ . Give your answer to 3 decimal places.

$$\begin{aligned} \text{a) } f(2) &= -1.9116\dots \\ f(3) &= 2.032\dots \end{aligned}$$

(5)

- change of sign
- $f(x)$  is continuous
- root  $\alpha$  exists between  $x=2$  &  $x=3$ .

$$\text{b) } x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 3x^2 - 6x - \frac{5}{4}x^{-3.5} \quad \text{when } x=3$$

$$f'(x) = 8.973\dots$$

$$3 - \frac{2.032075015}{8.973270821}$$

$$\alpha = 2.774$$

3. Given that  $z = x + iy$ , where  $x$  and  $y$  are real numbers, solve the equation

$$(z - 2i)(z^* - 2i) = 21 - 12i$$

where  $z^*$  is the complex conjugate of  $z$ .

(6)

$$z^* = x - iy$$

$$(x + iy - 2i)(x - iy - 2i)$$

$$x^2 - xiyi - 2xi + xiyi - 2xi + y^2 - 4$$

$$x^2 + y^2 - 4xi - 4$$

$$(x^2 + y^2 - 4) + i(-4x)$$

$$u + iv$$

$$x^2 + y^2 - 4 = 21$$

$$-4x = -12$$

$$4x = 12$$

$$x = 3$$

$$(3)^2 + y^2 - 4 = 21$$

$$9 + y^2 - 4 = 21$$

$$y^2 = 16 \quad y = \pm 4$$

$$z = 3 \pm 4i$$

4. The parabola  $C$  has cartesian equation  $y^2 = 12x$

The point  $P(3p^2, 6p)$  lies on  $C$ , where  $p \neq 0$

(a) Show that the equation of the normal to the curve  $C$  at the point  $P$  is

$$y + px = 6p + 3p^3$$

This normal crosses the curve  $C$  again at the point  $Q$ .

(5)

Given that  $p = 2$  and that  $S$  is the focus of the parabola, find

(b) the coordinates of the point  $Q$ ,

(5)

(c) the area of the triangle  $PQS$ .

(4)

$$a) \quad y = \sqrt{12x} = 2\sqrt{3}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \sqrt{3}x^{-\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{x}} \quad \text{when } x = 3p^2 \quad \frac{\sqrt{3}}{\sqrt{3p^2}} = \frac{1}{p} = m$$

at the normal  $m_n = -p$

$$y - y_1 = m(x - x_1)$$

$$y - 6p = -p(x - 3p^2)$$

$$y + px = 6p + 3p^3$$

$$b) \quad p = 2 \quad y + 2x = 12 + 24 \quad y^2 = 12x$$

$$y + 2x = 36 \quad \frac{y^2}{12} = x \quad \frac{y^2}{6} = 2x$$

$$y^2 + y^2 = 36$$

$$6y + y^2 = 216$$

$$y^2 + 6y - 216 = 0 \quad (y + 18)(y - 12) = 0$$

$$y = -18 \text{ or } 12$$

c) Focus is  $(3, 0)$

$$y = 0 \quad x = 18$$

$$\text{Area} = \frac{1}{2} \times (18 - 3) \times 12 + \frac{1}{2} (18 - 3)(18) = 225$$

5. The quadratic equation

$$4x^2 + 3x + 1 = 0$$

has roots  $\alpha$  and  $\beta$ .

(a) Write down the value of  $(\alpha + \beta)$  and the value of  $\alpha\beta$ .

(2)

(b) Find the value of  $(\alpha^2 + \beta^2)$ .

(2)

(c) Find a quadratic equation which has roots

$$(4\alpha - \beta) \text{ and } (4\beta - \alpha)$$

giving your answer in the form  $px^2 + qx + r = 0$  where  $p$ ,  $q$  and  $r$  are integers to be determined.

(4)

$$5) a) \quad \alpha + \beta = -\frac{3}{4} \quad \alpha\beta = \frac{1}{4}$$

$$b) \quad (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\left(-\frac{3}{4}\right)^2 - 2\left(\frac{1}{4}\right) = \frac{1}{16}$$

$$c) \text{ sum } (4\alpha - \beta) + (4\beta - \alpha) = 3(\alpha + \beta) = 3\left(-\frac{3}{4}\right) = -\frac{9}{4}$$

$$\text{product } (4\alpha - \beta)(4\beta - \alpha)$$

$$16\alpha\beta - 4\beta^2 - 4\alpha^2 + \alpha\beta = 17\alpha\beta - 4(\alpha^2 + \beta^2)$$

$$17\left(\frac{1}{4}\right) - 4\left(\frac{1}{16}\right) = 4$$

$$x^2 - \left(-\frac{9}{4}x\right) + 4 = 0$$

$$4x^2 + 9x + 16 = 0$$

6.

$$(i) \quad \mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix  $\mathbf{A}$ . (2)

(b) Describe fully the single transformation represented by the matrix  $\mathbf{B}$ . (2)

The transformation represented by  $\mathbf{A}$  followed by the transformation represented by  $\mathbf{B}$  is equivalent to the transformation represented by the matrix  $\mathbf{C}$ .

(c) Find  $\mathbf{C}$ . (2)

$$(ii) \quad \mathbf{M} = \begin{pmatrix} 2k+5 & -4 \\ 1 & k \end{pmatrix}, \text{ where } k \text{ is a real number.}$$

Show that  $\det \mathbf{M} \neq 0$  for all values of  $k$ . (4)

i) a) stretch scale factor 3 parallel to x axis

b) rotation 210 degrees anticlockwise about (0,0).

$$c) \quad \mathbf{C} = \mathbf{B}\mathbf{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} -\frac{3\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$ii) \quad \det \mathbf{M} = (2k+5)k - 1(-4) \\ = 2k^2 + 5k + 4$$

$$b^2 - 4ac \\ = 25 - 32$$

$$b^2 - 4ac < 0$$

no real roots  
so  $\det \mathbf{M} \neq 0$

7. Given that, for all positive integers  $n$ ,

$$\sum_{r=1}^n (r+a)(r+b) = \frac{1}{6}n(2n+1)(n-1)$$

where  $a$  and  $b$  are constants and  $a > b$ ,

- (a) find the value of  $a$  and the value of  $b$ .

(8)

- (b) Find the value of

$$\sum_{r=9}^{20} (r+a)(r+b)$$

$$7) a) (r+a)(r+b) = r^2 + ra + rb + ab \quad (3)$$

$$\sum r^2 + \sum r(a+b) + \sum ab$$

$$\frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1)(a+b) + abn = \frac{n}{6}(2n+1)(n-1)$$

$$\frac{n}{6} \left[ (n+1)(2n+1) + 3(n+1)(a+b) + 6ab \right] = \frac{n}{6}(2n+1)(n-1)$$

$$2n^2 + 3n + 1 + 3(n+1)(a+b) + 6ab = 2n^2 + 9n - 11$$

$$3n + 1 + 3na + 3a + 3nb + 3b + 6ab = 9n - 11$$

$$n(3 + 3a + 3b) + (3a + 3b + 1 + 6ab) = 9n - 11$$

$$n(1 + a + b) = 3n$$

$$1 + a + b = 3$$

$$a + b = 2$$

$$3a + 3b + 1 + 6ab = -11$$

$$3(2) + 1 + 6ab = -11$$

$$6ab = -3$$

$$a = 3 \quad b = -1$$

$$b) \sum_{r=9}^{20} (r+a)(r+b)$$

$$f(20) - f(8)$$

$$\frac{1}{6}(20)(51)(19) - \frac{1}{6}(8)(27)(7)$$

$$= 2978$$

8. (i) A sequence of numbers is defined by

$$u_1 = 5 \quad u_2 = 13$$

$$u_{n+2} = 5u_{n+1} - 6u_n \quad n \geq 1$$

Prove by induction that, for  $n \in \mathbb{Z}^+$ ,

$$u_n = 2^n + 3^n$$

(6)

(ii) Prove by induction that for  $n \geq 2$ , where  $n \in \mathbb{Z}$ ,

$$f(n) = 7^{2n} - 48n - 1$$

is divisible by 2304

8) i)  $n=1 \quad u_1 = 2^1 + 3^1 = 5$  true for  $n=1$  &  $n=2$  (6)  
 $n=2 \quad u_2 = 2^2 + 3^2 = 13$

assume  $u_k = 2^k + 3^k \quad u_{k+1} = 2^{k+1} + 3^{k+1}$

$$u_{k+2} = 5u_{k+1} - 6u_k = 5(2^{k+1} + 3^{k+1}) - 6(2^k + 3^k)$$

$$5(2^{k+1}) + 5(3^{k+1}) - 6(2^k) - 6(3^k)$$

$$5(2^{k+1}) + 5(3^{k+1}) - 3(2)(2^k) - 3(2)(3^k)$$

$$5(2^{k+1}) + 5(3^{k+1}) - 3(2^{k+1}) - 2(3^{k+1})$$

$$2^{(k+1)+1} + 3^{(k+1)+1} = 2^{k+2} + 3^{k+2}$$

if true for  $k$  &  $k+1$  then shown true  $k+2$   
 & as true for  $n=1$  &  $n=2$ , true  $n \in \mathbb{Z}^+$

ii)  $f(2) = 7^4 - 48(2) - 1 = 2304$  so true  $n=2$

Assume  $7^{2k} - 48k - 1 = 2304p$  integer  $p$ .

$$f(k+1) - f(k) = 7^{2k+2} - 48(k+1) - 1 - (7^{2k} - 48k - 1)$$

$$7^{2k+2} - 7^{2k} - 48 = 7^{2k}(49-1) - 48$$

$$48f(k) + 48^2k$$

$$48(2304p) + 2304k \quad f(k+1) = 49(2304p) + 2304k$$

if true  $k$ , shown true  $k+1$  and as true for  $n=2$   
 true for  $n \geq 2 \quad n \in \mathbb{Z}$