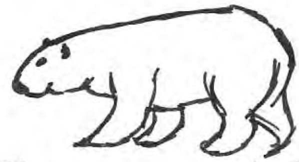
1. Given that k is a real number and that

$$\mathbf{A} = \begin{pmatrix} 1+k & k \\ k & 1-k \end{pmatrix}$$



find the exact values of k for which $\mathbf{A}$ is a singular matrix. Give your answers in their simplest form.

(3)

If $\text{Det}(A) = 0 \Rightarrow$ Singular

$\text{Det}(A) = (1+k)(1-k) - k^2 = 1 - 2k^2$

$\therefore$ If singular $2k^2 = 1 \quad \therefore k^2 = \frac{1}{2} \quad \therefore k = \pm\sqrt{\frac{1}{2}}$

$k = \pm\frac{\sqrt{2}}{2}$

2. $$f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, \qquad x > 0$$

(a) Find $f'(x)$.

(2)

The equation $f(x) = 0$ has a root α in the interval [12, 13].

(b) Using $x_0 = 12.5$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

(4)

a) $f'(x) = \frac{9}{2}x^{\frac{1}{2}} + \frac{25}{2}x^{-\frac{3}{2}}$

b) $x_0 = 12.5 \Rightarrow x_1 = 12.5 - \dfrac{f(12.5)}{f'(12.5)}$

$x_1 = 12.5 - \dfrac{3(12.5)^{\frac{3}{2}} - 25(12.5)^{-\frac{1}{2}} - 125}{\frac{9}{2}(12.5)^{\frac{1}{2}} + \frac{25}{2}(12.5)^{-\frac{3}{2}}}$

$\therefore x_1 = 12.468$

2

3. (a) Using the formula for $\sum_{r=1}^{n} r^2$ write down, in terms of n only, an expression for

$$\sum_{r=1}^{3n} r^2$$

(1)

(b) Show that, for all integers n, where $n > 0$

$$\sum_{r=2n+1}^{3n} r^2 = \frac{n}{6}(an^2 + bn + c)$$

where the values of the constants a, b and c are to be found.

(4)

a) $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\Rightarrow \sum_{r=1}^{3n} r^2 = \frac{1}{6}(3n)(3n+1)(6n+1)$

b) $\sum_{r=2n+1}^{3n} r^2 = \sum_{r=1}^{3n} r^2 - \sum_{r=1}^{2n} r^2$

$= \frac{1}{6}(3n)(3n+1)(6n+1) - \frac{1}{6}(2n)(2n+1)(4n+1)$

$= \frac{1}{6}n\left[(9n+3)(6n+1) - (4n+2)(4n+1)\right]$

$= \frac{1}{6}n\left[54n^2+27n+3-16n^2-12n-2\right]$

$= \frac{1}{6}n\left[38n^2+15n+1\right]$ $a=38$ $b=15$ $c=1$

4. $$z = \frac{4}{1+\mathrm{i}}$$

Find, in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$

(a) z

(2)

(b) z^2

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(c) find the value of p and the value of q.

(3)

a) $Z = \frac{4(1-i)}{(1+i)(1-i)} = \frac{4-4i}{1-i^2} = \frac{4-4i}{2} = 2-2i$

b) $z^2 = (2-2i)^2 = 4-8i+4i^2 = -8i$

c) roots α, β $\quad \alpha = 2-2i \Rightarrow \beta = 2+2i$

$x^2 - (\alpha+\beta)x + (\alpha\beta)$

$x^2 - 4x + (4-4i^2) \Rightarrow x^2 - 4x + 8$ $\quad p = -4$, $q = 8$

5. Points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p^2 \neq q^2$, lie on the parabola $y^2 = 4ax$.

(a) Show that the chord PQ has equation

$$y(p+q) = 2x + 2apq$$

(5)

Given that this chord passes through the focus of the parabola,

(b) show that $pq = -1$

(1)

(c) Using calculus find the gradient of the tangent to the parabola at P.

(2)

(d) Show that the tangent to the parabola at P and the tangent to the parabola at Q are perpendicular.

(2)

a) $M_{PQ} = \dfrac{2aq - 2ap}{aq^2 - ap^2} = \dfrac{2\cancel{a}\cancel{(q-p)}}{\cancel{a}\cancel{(q-p)}(q+p)} = \dfrac{2}{q+p}$

$y - y_1 = m(x - x_1) \Rightarrow y - 2ap = \dfrac{2}{q+p}(x - ap^2)$

$\Rightarrow y(q+p) - 2ap(q+p) = 2x - 2ap^2$

$\Rightarrow y(q+p) = 2x - \cancel{2ap^2} + 2apq + \cancel{2ap^2}$

$\therefore y(p+q) = 2x + 2apq$ #

b) chord passes through focus $(a, 0)$

$\Rightarrow 0(p+q) = 2a + 2apq$

$\Rightarrow -2a = 2apq \quad \therefore pq = \dfrac{-2a}{2a} \quad \therefore pq = -1$ #

c) $y^2 = 4ax \Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{4a}{2y}$

at P $y = 2ap$ $\therefore M_t = \frac{4a}{2(2ap)} = \frac{4\cancel{a}}{\cancel{4a}p} = \frac{1}{p}.$

d)

if perp $M_{t_p} \times M_{t_q} = -1$

$M_{t_p} = \frac{1}{p} \quad M_{t_q} = \frac{1}{q}$

$M_{t_p} \times M_{t_q} = \frac{1}{p} + \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1$

#

6.

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix **P**. **(2)**

The transformation U maps the point A, with coordinates (p, q), onto the point B, with coordinates $(6\sqrt{2}, 3\sqrt{2})$.

(b) Find the value of p and the value of q. **(3)**

The transformation V, represented by the 2×2 matrix **Q**, is a reflection in the line with equation $y = x$.

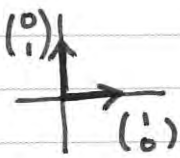
(c) Write down the matrix **Q**. **(1)**

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix **R**.

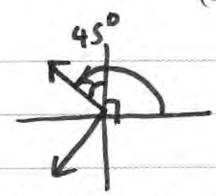
(d) Find the matrix **R**. **(3)**

(e) Deduce that the transformation T is self-inverse. **(1)**

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$\Rightarrow \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$



rotation about (0,0) through 135°.

b) $PA = B \Rightarrow A = P^{-1}(B)$

$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ $\quad \det(P) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

$\therefore P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} = \begin{pmatrix} -6+3 \\ -6-3 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection through $y=x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

y becomes x, x becomes y

d) $$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = R$$

e) $$\det(R) = \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} = -1$$

$$R^{-1} = -1 \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore R = R^{-1}$ $\therefore$ Self Inverse

?

7. A complex number z is given by

$$z = a + 2\text{i}$$

where a is a non-zero real number.

(a) Find $z^2 + 2z$ in the form $x + \text{i}y$ where x and y are real expressions in terms of a.
(4)

Given that $z^2 + 2z$ is real,

(b) find the value of a.
(1)

Using this value for a,

(c) find the values of the modulus and argument of z, giving the argument in radians, and giving your answers to 3 significant figures.
(3)

(d) Show the points P, Q and R, representing the complex numbers z, z^2 and $z^2 + 2z$ respectively, on a single Argand diagram with origin O.
(3)

(e) Describe fully the geometrical relationship between the line segments OP and QR.
(2)

a) $z^2 = (a+2i)^2 = a^2 + 4ai + 4i^2 = (a^2-4)+4ai$

$z^2 + 2z = (a^2-4)+4ai+2a+4i$

$= (a^2+2a-4)+(4a+4)i$

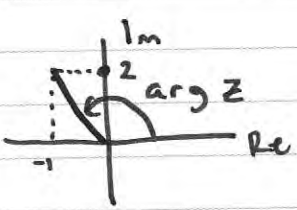
$x = a^2+2a-4$
$y = 4a+4.$

b) z^2+2z is Real $\Rightarrow \text{Im} = 0$ $\therefore 4a+4=0$

$a=-1$

2

c) $|z| = |-1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$

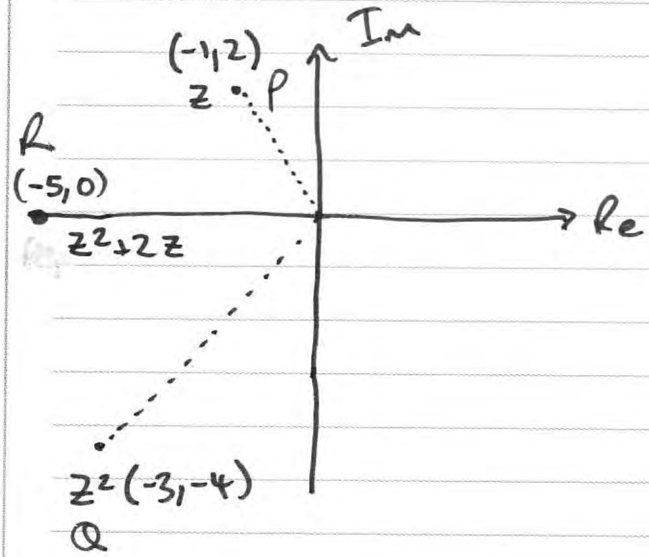


$\arg z = \pi - \tan^{-1}\left(\frac{2}{1}\right)$

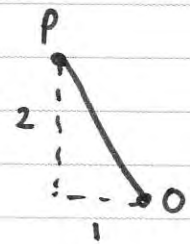
$= 2.03^c$

2

d) $z = -1 + 2i \quad z^2 = -3 - 4i \quad z^2 + 2z = -5$

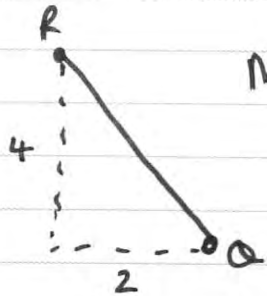


e) $\overrightarrow{OP}$



$M_{\overrightarrow{OP}} = -2 \qquad |\overrightarrow{OP}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\overrightarrow{QR}$



$M_{\overrightarrow{QR}} = -2 \qquad |\overrightarrow{QR}| = \sqrt{4^2 + 2^2} = \sqrt{20}$
$= 2\sqrt{5}$

$\overrightarrow{OP}$ and $\overrightarrow{QR}$ are parallel

$\overrightarrow{QR}$ is twice the length of $\overrightarrow{OP}$

$\overrightarrow{QR} = 2 \times \overrightarrow{OP}$

8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

(5)

(ii) A sequence of positive rational numbers is defined by

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3}u_n + \frac{8}{9}, \qquad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3}\right)^n + \frac{4}{3}$$

(5)

a) $n=1$ $\quad \sum_{1}^{1} \frac{2r+1}{r^2(r+1)^2} = \frac{3}{1(2)^2} = \frac{3}{4}$

$1 - \frac{1}{(1+1)^2} = 1 - \frac{1}{4} = \frac{3}{4}$ $\quad \therefore$ LHS = RHS
true for n=1

assume true for n=k.

$$\sum_{1}^{k} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$$

$n=k+1$ $\quad \sum_{1}^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_{1}^{k} \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2(k+2)^2}$

$$= \left(1 - \frac{1}{(k+1)^2}\right) + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \left(\frac{1}{(k+1)^2}\frac{(k+2)^2}{(k+2)^2} - \frac{2k+3}{(k+1)^2(k+2)^2}\right)$$

$$1 - \frac{k^2+4k+4-2k-3}{(k+1)^2(k+2)^2} = 1 - \frac{k^2+2k+1}{(k+1)^2(k+2)^2}$$

$$= 1 - \frac{(k+1)^2}{(k+1)^2(k+2)^2} = 1 - \frac{1}{(k+2)^2}$$

$n=k+1$ RHS $= 1 - \frac{1}{(k+1+1)^2} = 1 - \frac{1}{(k+2)^2}$

$\therefore$ LHS = RHS $\therefore$ true for $n=k+1$

true for $n=1$, if true for $n=k$ then true for $n=k+1$
$\therefore$ by Mathematical Induction true for all $n \in \mathbb{Z}^+$

(ii) $u_1 = 3$ $u_1 = 5\left(\frac{1}{3}\right)^1 + \frac{4}{3} = \frac{5}{3} + \frac{4}{3} = 3$

LHS = RHS $\therefore$ true for $n=1$

$u_2 = \frac{1}{3}(3) + \frac{8}{9} = \frac{17}{9}$

$u_2 = 5\left(\frac{1}{3}\right)^2 + \frac{4}{3} = \frac{5}{9} + \frac{12}{9} = \frac{17}{9}$

LHS = RHS $\therefore$ true for $n=2$

assume true for $n=k \Rightarrow u_k = 5\left(\frac{1}{3}\right)^k + \frac{4}{3}$

$n=k+1$ $u_{k+1} = \frac{1}{3}u_k + \frac{8}{9} = \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{4}{9} + \frac{8}{9}$

$$= \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{12}{9} = 5\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^k + \frac{4}{3}$$

$$= 5\left(\frac{1}{3}\right)^{k+1} + \frac{4}{3}$$

RHS $u_{k+1} = 5\left(\frac{1}{3}\right)^{k+1} + \frac{4}{3}$

$\therefore$ RHS = LHS $\therefore$ true for $n = k+1$

true for $n=1, n=2$ true for $n=k+1$ if true for $n=k$ $\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}^+$.

9. The rectangular hyperbola, H, has cartesian equation $xy = 25$

(a) Show that an equation of the normal to H at the point $P\left(5p, \frac{5}{p}\right)$, $p \neq 0$, is

$$y - p^2x = \frac{5}{p} - 5p^3$$

(5)

This normal meets the line with equation $y = -x$ at the point A.

(b) Show that the coordinates of A are

$$\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right)$$

(3)

The point M is the midpoint of the line segment AP.
Given that M lies on the positive x-axis,

(c) find the exact value of the x coordinate of point M.

(3)

a) $y = 25x^{-1} \Rightarrow \frac{dy}{dx} = -25x^{-2} = \frac{-25}{x^2}$

$\Rightarrow$ at P $\quad M_t = \frac{-25}{25p^2} = \frac{-1}{p^2}$

$M_n = p^2$

$y - y_1 = m(x - x_1)$

$y - \frac{5}{p} = p^2(x - 5p) \Rightarrow y - \frac{5}{p} = p^2x - 5p^3$

$\therefore y - p^2x = \frac{5}{p} - 5p^3$

b) $y = -x \Rightarrow -x - p^2x = \frac{5}{p} - 5p^3$

$\Rightarrow x + p^2x = 5p^3 - \frac{5}{p} = \frac{5p^4 - 5}{p}$

$\Rightarrow x(1+p^2) = \frac{5(p^4-1)}{p}$

$\Rightarrow x = \frac{5(\cancel{p^2+1})(p^2-1)}{p(\cancel{p^2+1})} = \frac{5p^2-5}{p}$

$\therefore x = 5p - \frac{5}{p} \quad y = \frac{5}{p} - 5p \qquad \left(5p - \frac{5}{p}, \frac{5}{p} - 5p\right)$ #

c) Midpoint AP

$$= \left(\frac{5p - \frac{5}{p} + 5p}{2}, \frac{\frac{5}{p} + \frac{5}{p} - 5p}{2}\right)$$

$$= \left(5p - \frac{5}{2p}, \frac{5}{p} - \frac{5p}{2}\right)$$

M lies on x-axis $\therefore y = 0$

$\frac{5}{p} = \frac{5p}{2} \Rightarrow \quad 10 = 5p^2 \quad \therefore p^2 = 2$

$\therefore p = \sqrt{2}$

(not $-\sqrt{2}$ as it is on the x-axis)

$x = 5\sqrt{2} - \frac{5}{2\sqrt{2}} = 5\sqrt{2} - \frac{5\sqrt{2}}{4} = \frac{15\sqrt{2}}{4}$