F1 Jan 2017 (MA)

Q1a) $f(2) = 2^2 - 10\sin(2) - 2 = -7.09297...$

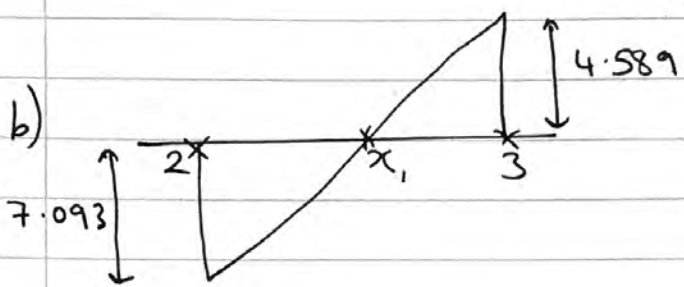
$f(3) = 2^3 - 10\sin(3) - 2 = 4.5888...$

change in sign over the interval $[2, 3]$

$\therefore$ a root lies between $x = 2$ and $x = 3$.

remember to put your calc in radians

b)



$$\frac{3 - x_1}{x_1 - 2} = \frac{4.589}{7.093}$$

let $\frac{4.589}{7.093} = c$,

then $\frac{3 - x_1}{x_1 - 2} = c$

$$c x_1 - 2c = 3 - x_1$$

$$x_1(c+1) = 3 + 2c$$

$$x_1 = \frac{3 + 2c}{c+1} = \boxed{2.607}$$

Q2a) $\boxed{\alpha + \beta = \frac{1}{2}}$, $\boxed{\alpha\beta = \frac{3}{2}}$ $\left[x^2 - \frac{x}{2} + \frac{3}{2}\right] = 0$

b) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta}{\alpha\beta} + \frac{\alpha}{\beta\alpha} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\frac{1}{2}}{\frac{3}{2}} = \boxed{\frac{1}{3}}$

c) $\left(x - \left(2\alpha - \frac{1}{\beta}\right)\right)\left(x - \left(2\beta - \frac{1}{\alpha}\right)\right) = 0$ //

$\Rightarrow x^2 - x\left(2\beta - \frac{1}{\alpha}\right) - x\left(2\alpha - \frac{1}{\beta}\right) + \left(2\beta - \frac{1}{\alpha}\right)\left(2\alpha - \frac{1}{\beta}\right) = 0$

$\Rightarrow x^2 - x\left(2\beta + 2\alpha - \frac{1}{\alpha} - \frac{1}{\beta}\right) + 4\alpha\beta - 2 - 2 + \frac{1}{\alpha\beta} = 0$

$\Rightarrow x^2 - x\left(2(\alpha+\beta) - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right) + 4(\alpha\beta) - 4 + \frac{1}{(\alpha\beta)} = 0$

$\Rightarrow x^2 - x\left(2 \cdot \frac{1}{2} - \frac{1}{3}\right) + 4 \cdot \frac{3}{2} - 4 + \frac{1}{\frac{3}{2}} = 0$

$\Rightarrow x^2 - x\left(\frac{2}{3}\right) + \frac{8}{3} = 0$

$\xrightarrow{\times 3} \boxed{3x^2 - 2x + 8 = 0}$

Q3.) $x = -1-3i$ will also be a root.

$\therefore f(x) = (x-(-1-3i))(x-(-1+3i))(x^2+ax+b) = 0$

$\Rightarrow \left[x^2 - x(-1+3i-1-3i) + (-1-3i)(-1+3i)\right](x^2+ax+b) = 0$ //

$\Rightarrow [x^2 + 2x + 10][x^2 + ax + b] = 0$

$\Rightarrow x^4 + ax^3 + bx^2 + 2x^3 + 2x^2a + 2bx + 10x^2 + 10ax + 10b = 0$

$\Rightarrow x^4 + x^3(a+2) + x^2(b+2a+10) + x(2b+10a) + 10b = 0$ //

compare with given eqn : $10b = 160$

$\therefore b = 16$ //

$a + 2 = 2$

$\therefore a = 0$ //

so $f(x) = (x^2+16)(x^2+2x+10) = 0$ //

$x^2 + 16 = 0$
$x^2 = -16$
$x = \pm 4i$

all solutions : $x = -1 + 3i$
$x = -1 - 3i$
$x = 4i$
$x = -4i$

Q4a) $\sum_{r=1}^{n} r(2r+1)(3r+1) = \sum_{r=1}^{n} r(6r^2 + 5r + 1)$

$= \sum_{r=1}^{n} 6r^3 + 5r^2 + r = 6\sum_{r=1}^{n} r^3 + 5\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$= 6\left[\frac{n^2}{4}(n+1)^2\right] + 5\left[\frac{n}{6}(n+1)(2n+1)\right] + \frac{n}{2}(n+1)$

$= \frac{3}{2}n^2(n+1)^2 + \frac{5n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1)$

$= \frac{n}{6}(n+1)\left[9n(n+1) + 5(2n+1) + 3\right]$

$= \frac{n}{6}(n+1)\left[9n^2 + 9n + 10n + 5 + 3\right]$

$= \frac{n}{6}(n+1)(9n^2 + 19n + 8)$

$a = 9$
$b = 19$
$c = 8$

b) $\sum_{r=1}^{20} r(2r+1)(3r+1) - \sum_{r=1}^{9} r(2r+1)(3r+1)$

$= \frac{20}{6}(21)(9(20)^2 + 19(20) + 8) - \frac{9}{6}(10)(9(9^2) + 19(9) + 8)$

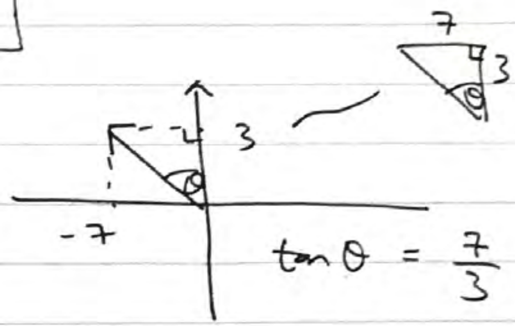
$= 279160 - 13620 = \boxed{265540}$

Q5a) $z = -7 + 3i$

$|z| = \sqrt{7^2 + 3^2} = \boxed{\sqrt{58}}$

b) $\arg z = \frac{\pi}{2} + \tan^{-1}\left(\frac{7}{3}\right)$

$= \boxed{2.74^c}$



$\tan\theta = \frac{7}{3}$

c) $w = 3 - 6i - \frac{z}{1+i}$

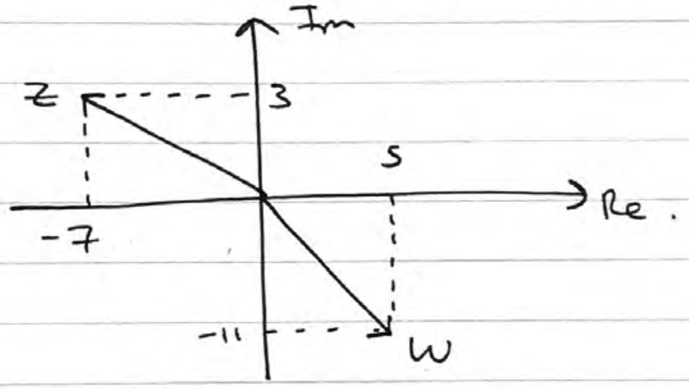
$w = 3 - 6i - \frac{(-7+3i)}{1+i}$

$w = 3 - 6i - \left[\frac{(-7+3i)(1-i)}{(1+i)(1-i)}\right]$

$w = 3 - 6i - \left[\frac{-7 + 7i + 3i + 3}{2}\right] = 3 - 6i - \left[\frac{-4+10i}{2}\right]$

$w = 3 - 6i - [-2 + 5i] = \boxed{5 - 11i}$

d)



Q6a) $f(x) = x^3 - \frac{1}{2}x^{-1} + x^{\frac{3}{2}}$

$f'(x) = 3x^2 + \frac{1}{2}x^{-2} + \frac{3}{2}x^{\frac{1}{2}}$

$f(0.6) = -0.152575\ldots$

$f'(0.6) = 3.630784\ldots$

$\therefore \alpha \simeq 0.6 - \frac{(-0.152575)}{3.630784} \simeq \boxed{0.642}$

b) $f(0.6415) = (0.6415)^3 - \frac{1}{2(0.6415)} + (0.6415)^{\frac{3}{2}} = (-1.63 \times 10^{-3})$

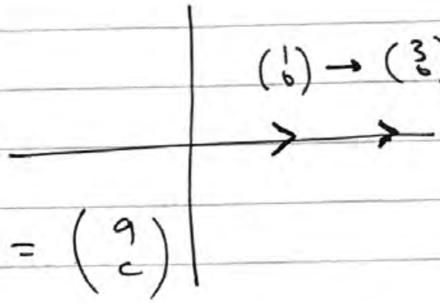
$f(0.6425) = (0.6425)^3 - \frac{1}{2(0.6425)} + (0.6425)^{\frac{3}{2}} = (2.02 \times 10^{-3})$

change in sign between $x = 0.6415$ and $x = 0.6425$ $\therefore$ answer to (a) is correct to 3dp.

Q7ai) $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection in the y-axis

b) $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

$\therefore B = \boxed{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}$

ii a) $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k\cos\theta & -k\sin\theta \\ k\sin\theta & k\cos\theta \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & -4 \end{pmatrix}$

$k\cos\theta = -4 \sim ①$

$k\sin\theta = -3 \sim ②$

$①^2 + ②^2 : k^2\cos^2\theta + k^2\sin^2\theta = 3^2 + 4^2$

$k^2(\sin^2\theta + \cos^2\theta) = 25$

$\therefore k^2 = 25$

$\therefore k = \sqrt{25} = 5$ $(k > 0)$

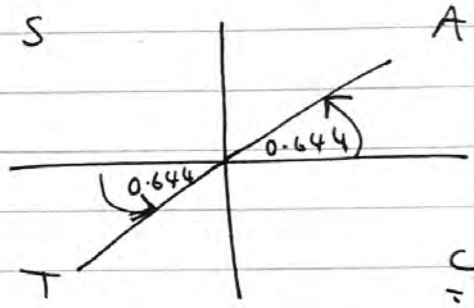
b) from (a), $5\cos\theta = -4$ $\quad 5\sin\theta = -3$

① $\cos\theta = -\frac{4}{5}$ $\quad \sin\theta = -\frac{3}{5}$ ②

we want the value of θ where both of these are true.

$\frac{②}{①}$: $\tan\theta = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4}$

$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.644^c, \pi + 0.644^c$



$\therefore \theta = 0.644^c, \pi + 0.644^c$

Reject $\theta = 0.644$ because we want the value of θ that is in the 3rd quadrant

as this is the only quadrant where both $[\sin\theta = -\frac{3}{5}]$ and $[\cos\theta = \frac{-4}{5}]$ can be true.

$\therefore \theta = \pi + 0.644$
$= \boxed{3.79^c}$

c) $\det M = (-4)^2 - 3(-3) = 25$

$\therefore M^{-1} = \frac{1}{25}\begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$

Q8a) $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2at} = \frac{1}{t}$ //

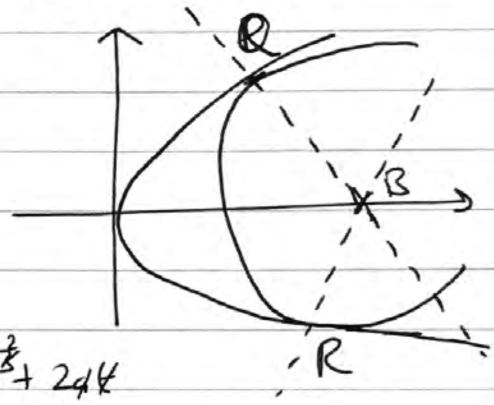
$\therefore$ at normal, $m = -t$ // $\quad (-t \times \frac{1}{t} = -1)$

$y - 2at = -t(x - at^2)$

$y = -xt + at^3 + 2at \quad \therefore \boxed{y + tx = at^3 + 2at}$

b) $OS = a \quad \therefore OB = 5a \quad \therefore B\,[5a, 0]$

c) The normals at Q/R will pass through B.



$\Rightarrow y + tx = at^3 + 2at$

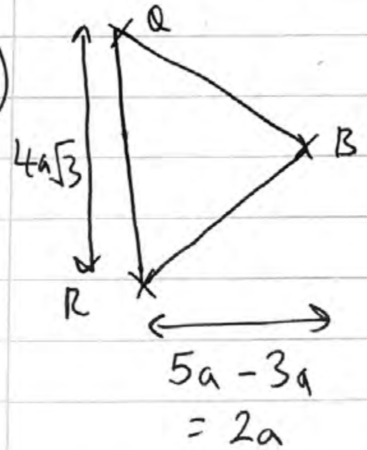
substitute $x = 5a,\ y = 0$: $\cancel{t}(5\cancel{a}) = \cancel{a}t^{\cancel{3}2} + 2\cancel{a}\cancel{t}$

$5 = t^2 + 2$

$\therefore t^2 = 3 \quad \therefore t = \pm\sqrt{3}$ //

$t = \sqrt{3} \longrightarrow Q\,(a(3), 2a\sqrt{3})$ and $R\,(a(3), -2a\sqrt{3})$

d)



Area $\Delta BQR = \frac{1}{2} \times 4a\sqrt{3} \times 2a$

$= \boxed{4a^2\sqrt{3}}$

we want to prove: $(k+1)^3(k+2) = $ sum to $(k+1)$

(Q9i) $\displaystyle\sum_{r=1}^{n}(4r^3-3r^2+r) = n^3(n+1)$

$n=1$: LHS $= 4-3+1 = 2 =$ RHS $= 1^3(2) = 2$

$\therefore$ relationship is true for $n=1$.

assume true for $n=k$,

ie $\left[\displaystyle\sum_{r=1}^{k} 4r^3-3r^2+r = k^3(k+1)\right]$

now consider $n=k+1$,

$$\sum_{r=1}^{k+1} 4r^3+(-3r^2)+r = k^3(k+1) + 4(k+1)^3 - 3(k+1)^2 + k+1$$

$$= k^3(k+1)+4(k+1)^3-3(k+1)^2+(k+1)$$

$$= (k+1)^3\left[\frac{k^3}{(k+1)^2}+4-\frac{3}{(k+1)}+\frac{1}{(k+1)^2}\right]$$

$$= (k+1)^3\left[\frac{k^3+4(k+1)^2-3(k+1)+1}{(k+1)^2}\right]$$

$$= (k+1)^3\left[\frac{k^3+4k^2+8k+4-3k-3+1}{(k+1)^2}\right]$$

$$= (k+1)^3\left[\frac{k^3+4k^2+5k+2}{(k+1)^2}\right]$$

$$= (k+1)^3\left[\frac{(k+1)^2(k+2)}{(k+1)^2}\right] = \boxed{(k+1)^3(k+2)}$$

$\therefore$ true for $n=k+1$.

$\therefore$ true for $n=1$

$\therefore$ true for $n=k+1$ when $(n=k)$ is assumed true.

$\therefore$ by Mathematical Induction true for all $n \in \mathbb{Z}^+$

ii) $f(n) = 5^{2n} + 3n - 1$

$n=1$: $f(1) = 5^2 + 3 - 1 = 27 = (9) \times 3$

$\therefore$ true for $n=1$.

assume $\left[f(k) = 5^{2k} + 3k - 1\right]$ is div. by 9.

consider $f(k+1)$,

$f(k+1) = 5^{2(k+1)} + 3(k+1) - 1$

$f(k+1) = 5^{2k+2} + 3k + 3 - 1$

factor out $f(k)$

$f(k+1) = \left[25(5^{2k}) + 3k - 1\right] + 3$

$f(k+1) = 25\left[5^{2k} + 3k - 1\right] - 24(3k) + 24(1) + 3$

$f(k+1) = 25f(k) - 72k + 27$

$f(k+1) = 25f(k) - 9 \times 8k + 9 \times 3$

$\therefore$ true for $n = k+1$.

$\therefore$ true for $n=1$

$\therefore$ true for $n=k+1$ when assumed true for $n=k$.

$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$