

F1 January 2018 (MA)

$$Q1a) f(x) = 3x^2 - \frac{5}{3}x^{-\frac{1}{2}} - 6$$

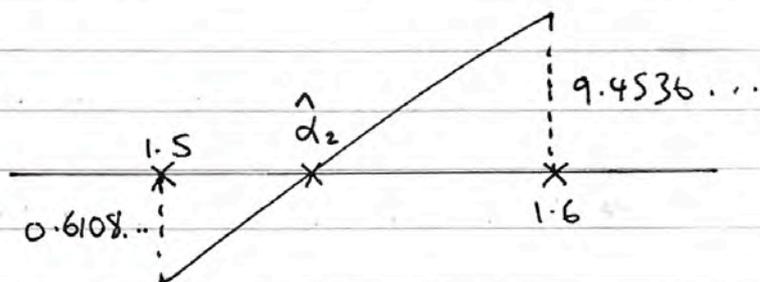
$$f'(x) = 6x + \frac{5}{6}x^{-\frac{3}{2}}$$

$$f(1.5) = -0.6108\dots$$

$$f'(1.5) = 9.4536\dots$$

$$x_1 = 1.5 - \frac{-0.6108\dots}{9.4536\dots} = \boxed{1.565} = 2, \text{ to 3 dp.}$$

b)



$$\frac{1.6 - \hat{a}}{2 - 1.5} = \frac{9.4536\dots}{0.6108\dots}$$

$$\text{let } \frac{9.4536\dots}{0.6108\dots} = c,$$

$$\text{then } \frac{1.6 - \hat{a}}{\hat{a} - 1.5} = c$$

$$2c - 1.5c = 1.6 - \hat{a}$$

$$\hat{a}(c+1) = 1.6 + 1.5c \quad \therefore \hat{a} = \frac{1.6 + 1.5c}{c+1}$$

$$\therefore \lambda = \boxed{1.563} \text{ to 3 d.p.}$$

(Q2a) If $2+3i$ is a root then $\boxed{2-3i}$ is also a root. (complex conjugate pair).

$$\begin{aligned} \therefore f(z) &= (z - (2+3i))(z - (2-3i))(z^2 + az + b) = 0 \\ &= (z^2 - z(2+2) + (2-3i)(2+3i))(z^2 + az + b) \\ &= (z^2 - 4z + 13)(z^2 + az + b) = 0 \end{aligned}$$

$$\begin{aligned} \therefore f(z) &= z^4 + az^3 + bz^2 - 4z^3 - 4az^2 - 4bz \\ &\quad + 13z^2 + 13az + 13b = 0 \end{aligned}$$

$$\begin{aligned} &= z^4 + (a-4)z^3 + (b-4a)z^2 + (13)z^2 \\ &\quad + (13a-4b)z + 13b = 0 \end{aligned}$$

compare coefficients with given eqn,

$$\begin{aligned} \underline{z^3} : \quad a - 4 &= -6 \\ \therefore a &= -2 \end{aligned}$$

$$\begin{aligned} \underline{z} : \quad -94 &= 13a - 4b \\ \therefore b &= \frac{13(-2) + 94}{4} = 17 \end{aligned}$$

$$\therefore f(z) = (z^2 - 4z + 13)(z^2 - 2z + 17) = 0$$

$$z^2 - 2z + 17 = 0$$

By Quadratic formula: $z = \frac{2 \pm \sqrt{4 - 4(17)}}{2}$

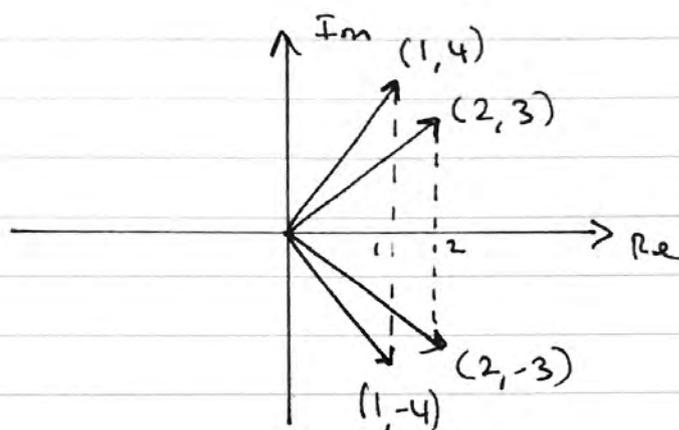
$$a = 1, b = -2, c = 17$$

$$z = \frac{2 \pm 8i}{2} = 1 \pm 4i //$$

so all roots :

$$\boxed{\begin{array}{l} z = 2 + 3i \\ z = 2 - 3i \\ z = 1 + 4i \\ z = 1 - 4i \end{array}}$$

b)



$$Q3a) \sum_1^n r^2(r+1) = \sum_1^n r^3 + r^2 = \sum_1^n r^3 + \sum_1^n r^2$$

$$= \frac{n^2}{4} (n+1)^2 + \frac{n}{6} (n+1) (2n+1)$$

$$= \frac{n}{12} (n+1) [3n(n+1) + 2(2n+1)]$$

$$= \frac{n}{12} (n+1) [3n^2 + 3n + 4n + 2]$$

$$= \frac{n}{12} (n+1) [3n^2 + 7n + 2]$$

$$= \frac{n}{12} (n+1) (3n^2 + 7n + 2) // = \boxed{\frac{n}{12} (n+1)(n+2)(3n+1)}$$

b) $\sum_1^u 3^r$ is a geometric series, where

$$a = 3$$

$$r = 3$$

$$\therefore S_u = \frac{a(1-r^u)}{1-r} = \frac{3(1-3^u)}{-2} //$$

$$\sum_5^{25} r^2(r+1) = \sum_1^{25} r^2(r+1) - \sum_1^4 r^2(r+1)$$

$$\sum_5^{25} r^2(r+1) = \frac{25(26)(27)(3(25)+1)}{12} - \frac{4(5)(6)(3(4)+1)}{2}$$

$$= 11150 - 130$$

$$= 11020 //$$

$$\therefore 11020 + \frac{3(1-3^u)}{-2} = 140543$$

$$-\frac{3}{2}(1-3^u) = 29523$$

$$1-3^u = -19682$$

$$3^u = 19683$$

$$\log(3^u) = \log(19683)$$

$$u \log 3 = \log(19683)$$

$$\therefore u = \frac{\log 19683}{\log 3} = \boxed{9}$$

● Q4a) $3x^2 + 2x + 5 = 0$

$$\div 3 : x^2 + \frac{2}{3}x + \frac{5}{3} = 0$$

$$\therefore \alpha + \beta = -\frac{2}{3}$$

$$\& \quad \alpha\beta = \frac{5}{3}$$

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= \left(-\frac{2}{3}\right)^2 - 2\left(\frac{5}{3}\right) \\ &= \boxed{\frac{-26}{9}} \end{aligned}$$

b) $\alpha^3 + \beta^3 = ?$

$$(\alpha + \beta)^3 = (\alpha^2 + \beta^2 + 2\alpha\beta)(\alpha + \beta) = \alpha^3 + \alpha^2\beta + \beta^2\alpha + \beta^3 + 2\alpha^2\beta + 2\alpha\beta^2$$

$$\therefore (\alpha + \beta)^3 = \alpha^3 + \beta^3 + \alpha\beta(\alpha + \beta) + 2\alpha\beta(\alpha + \beta)$$

$$\text{so } (\alpha + \beta)^3 - \alpha\beta(\alpha + \beta) - 2\alpha\beta(\alpha + \beta) = \alpha^3 + \beta^3$$

$$\Rightarrow \alpha^3 + \beta^3 = \left(-\frac{2}{3}\right)^3 - \frac{5}{3}\left(-\frac{2}{3}\right) - 2\left(\frac{5}{3}\right)\left(-\frac{2}{3}\right)$$

$$\alpha^3 + \beta^3 = \boxed{\frac{82}{27}}$$

$$c) \left(x - \left(\alpha + \frac{\alpha}{\beta^2} \right) \right) \left(x - \left(\beta + \frac{\beta}{\alpha^2} \right) \right) = 0$$

$$x^2 - x \left(\beta + \alpha + \frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} \right) + \left(\alpha + \frac{\alpha}{\beta^2} \right) \left(\beta + \frac{\beta}{\alpha^2} \right) = 0$$

$$x^2 - x \left(\alpha + \beta + \frac{\alpha^3 + \beta^3}{\alpha^2 \beta^2} \right) + \alpha \beta + \frac{\beta}{\alpha} + \frac{\alpha}{\beta} + \frac{\alpha \beta}{(\alpha \beta)^2}$$

$$x^2 - x \left(\frac{-2}{3} + \frac{\frac{82}{27}}{\left(\frac{5}{3}\right)^2} \right) + \frac{5}{3} + \frac{\beta^2 + \alpha^2}{\alpha \beta} + \frac{\alpha \beta}{(\alpha \beta)^2}$$

$$x^2 - x \left(\frac{32}{75} \right) + \frac{5}{3} + \frac{-26}{9} + \frac{\frac{5}{3}}{\left(\frac{5}{3}\right)^2}$$

$$x^2 - \frac{32}{75}x + \frac{8}{15} = 0$$

$$\underline{\times 75} : \boxed{75x^2 - 32x + 40 = 0}$$

$$\bullet \text{ (Q5i)} \quad \frac{2z + 3}{z + 5 - 2i} = 1 + i$$

$$(z + 5 - 2i)(1 + i) = 2z + 3$$

$$z + zi + 5 + 5i - 2i + 2 = 2z + 3$$

$$z - zi = 2 + 2 + 3i$$

$$z(1 - i) = 4 + 3i$$

$$z = \frac{4 + 3i}{1 - i} = \frac{(4 + 3i)(1 + i)}{(1 - i)(1 + i)}$$

$$z = \frac{4 + 4i + 3i + 3i^2}{1 + 1} = \frac{1 + 7i}{2}$$

$$\therefore z = \boxed{\frac{1}{2} + \frac{7}{2}i}$$

$$\text{ii)} \quad w = (3 + \lambda i)(2 + i)$$

$$w = 6 + 3i + 2\lambda i - \lambda$$

$$w = (6 - \lambda) + (3 + 2\lambda)i$$

$$|w| = \sqrt{(6 - \lambda)^2 + (3 + 2\lambda)^2} = 15 //$$

$$\therefore (6 - \lambda)^2 + (3 + 2\lambda)^2 = 15^2$$

$$36 - 12\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 = 225$$

$$\Rightarrow 5\lambda^2 = 225 - 45$$

$$\Rightarrow 5\lambda^2 = 180$$

$$\Rightarrow \lambda^2 = 36$$

$$\therefore \boxed{\lambda = 6, \lambda = -6}$$

Q6a) $y^2 = 32x$

$$4a = 32$$

$$a = \frac{32}{4} = 8$$

$$\therefore \boxed{S(8, 0)}$$

b) $8 + 2 = \boxed{10}$

c) $2y \frac{dy}{dx} = 32$

↪ Implicit differentiation

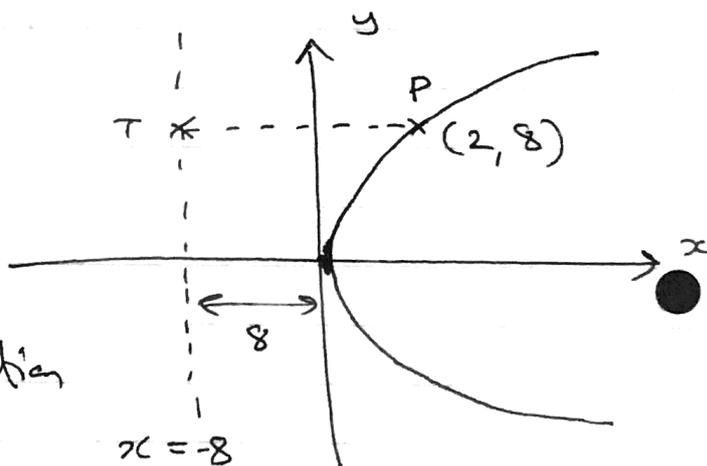
$$\therefore \frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y} =$$

$$= \frac{16}{8} = \underline{\underline{2}} \text{ at } P$$

$$\therefore y - 8 = 2(x - 2)$$

$$y = 2x - 4 + 8$$

hence $\boxed{y = 2x + 4}$



$$d) \quad y = 2x + 4 \text{ meets } xy = 4$$

$$\underline{\text{substituting } y} : x(2x + 4) = 4$$

$$2x^2 + 4x = 4$$

$$2x^2 + 4x - 4 = 0$$

$$x^2 + 2x - 2 = 0$$

By Quadratic Formula :

$$\left. \begin{array}{l} a = 1 \\ b = 2 \\ c = -2 \end{array} \right\} x = -1 \pm \sqrt{3} //$$

$$\begin{aligned} \text{at } x = -1 + \sqrt{3}, \quad y &= 2(-1 + \sqrt{3}) + 4 \\ &= 2 + 2\sqrt{3} // \end{aligned}$$

$$\begin{aligned} \text{at } x = -1 - \sqrt{3}, \quad y &= 2(-1 - \sqrt{3}) + 4 \\ &= 2 - 2\sqrt{3} // \end{aligned}$$

$$\text{so } L(-1 + \sqrt{3}, 2 + 2\sqrt{3})$$

$$M(-1 - \sqrt{3}, 2 - 2\sqrt{3})$$

$$\text{Q7ia)} \quad A = \begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix}$$

$$\det A = 6(-4) + 3u = 3u - 24$$

$$\therefore A^{-1} = \frac{1}{3u-24} \begin{pmatrix} -4 & -u \\ 3 & 6 \end{pmatrix}$$

$$\text{b)} \quad A^2 = \begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 36 - 3u & 6u - 4u \\ -18 + 12 & 16 - 3u \end{pmatrix}$$

$$= \begin{pmatrix} 36 - 3u & 2u \\ -6 & 16 - 3u \end{pmatrix}$$

$$c) A^2 + 3A^{-1} = \begin{pmatrix} 5 & 9 \\ -3 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 36-3k & 2k \\ -6 & 16-3k \end{pmatrix} + \begin{pmatrix} \frac{-4}{k-8} & \frac{-k}{k-8} \\ \frac{3}{k-8} & \frac{6}{k-8} \end{pmatrix}$$

using bottom left elements,

$$-6 + \frac{3}{k-8} = -3$$

$$\frac{3}{k-8} = 3$$

$$\therefore 1 = k-8 \quad \text{so } \boxed{k=9}$$

$$ii) M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

AC rotation through angle θ about O

one way stretch of s.f. p parallel to y -axis.

$$M = \begin{pmatrix} \cos \theta & -p \sin \theta \\ \sin \theta & p \cos \theta \end{pmatrix} //$$

$$\cos \theta = -\frac{1}{2} \quad \text{--- (1)}$$

$$p \sin \theta = \sqrt{3} \quad \text{--- (3)}$$

$$\sin \theta = \frac{\sqrt{3}}{2} \quad \text{--- (2)}$$

$$p \cos \theta = -1 \quad \text{--- (4)}$$

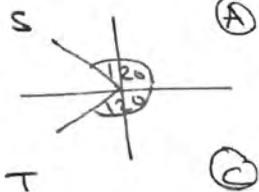
$$\textcircled{3}^2 + \textcircled{4}^2 : p^2 \sin^2 \theta + p^2 \cos^2 \theta = 3 + 1$$

$$p^2 (\sin^2 \theta + \cos^2 \theta) = 4$$

$$\therefore p^2 = 4 \rightarrow \boxed{p=2} \quad (p > 0)$$

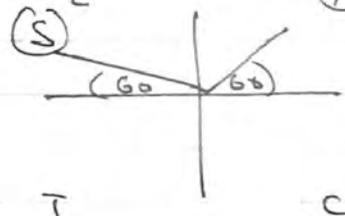
ii b)

$$\cos \theta = -\frac{1}{2} \quad \text{--- (A)}$$



$$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120^\circ, 240^\circ$$

$$\sin \theta = \frac{\sqrt{3}}{2} \quad \text{--- (A)}$$



$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ, 120^\circ$$

We want where both $\sin \theta = \frac{\sqrt{3}}{2}$ and $\cos \theta = -\frac{1}{2}$.

So the only valid value of θ is $\boxed{120^\circ}$ since this is a solution to both $\cos \theta = -\frac{1}{2}$ and $\sin \theta = \frac{\sqrt{3}}{2}$.

$$\bullet \text{ ii) } f(n) = 3^{2n+3} + 40n - 27$$

$$\underline{n=1}: f(1) = 3^5 + 40 - 27 = 256 \\ = (64) \times 4$$

\therefore true for $n=1$. //

assume true for $n=k$,

$$\left(\text{i.e. } f(k) = 3^{2k+3} + 40k - 27 \right. \\ \left. \text{is div. by } 64 \right)$$

consider $n=k+1$,

$$f(k+1) = 3^{2(k+1)+3} + 40(k+1) - 27 \\ = 3^{2k+3+2} + 40k - 27 + 40$$

$$= 9(3^{2k+3}) + 40k - 27 + 40$$

$$= \left[3^{2k+3} + 40k - 27 \right] \left[9 \right] + 40 - 8(40k) \\ + 8(27)$$

$$= 9f(k) + 40 - 320k + 216$$

$$= 9f(k) - 320k + 256$$

$$= \underbrace{9f(k)}_{\text{divisible by } 64} - 5k \underbrace{(64)}_{\text{divisible by } 64} + 64 \underbrace{(4)}_{\text{divisible by } 64}$$

\therefore true for $n=k+1$.

So ... true for $n=1$.
true for $n=k+1$ when
assumed true for $n=k$.

\therefore By Mathematical Induction
true for all $n \in \mathbb{Z}^+$
