F1 June 2018 (MA)

Q1) $\sum_{1}^{n} r(r+3) = \sum_{1}^{n} r^2 + 3r = \sum_{1}^{n} r^2 + 3\sum_{1}^{n} r$

$= \frac{n}{6}(n+1)(2n+1) + \frac{3n}{2}(n+1)$

$= \frac{n}{6}(n+1)\left[2n+1+9\right] = \frac{n}{6}(n+1)(2n+10)$

$= \frac{2n}{6}(n+1)(n+5) = \boxed{\frac{n}{3}(n+1)(n+5)}$

Q2a) $P = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

b) Enlargement by scale factor $u\sqrt{2}$ about $(0,0)$.

c) $PQ = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} u\sqrt{2} & 0 \\ 0 & u\sqrt{2} \end{pmatrix} = \begin{pmatrix} u & -u \\ u & u \end{pmatrix}$ //

$\text{Area}_{\text{image}} = \text{Area}_{\text{object}} \times \det|PQ|$.

$147 = 6 \times |\det PQ|$

$\therefore |\det PQ| = \frac{147}{6}$

$k^2 - (-k^2) = 2k^2 = |\det PQ|$ //

$\therefore \quad 2k^2 = \frac{147}{6}$

$k^2 = \frac{147}{12}$

$k = \pm\sqrt{\frac{147}{12}} = \pm\frac{7}{2}$

$k > 0 \quad \therefore \boxed{k = \frac{7}{2}}$

Q3a) $y^2 = \underset{4}{6}x$

$4a = 6$

$a = \frac{6}{4} = \frac{3}{2}$

$\therefore \boxed{S\left(\frac{3}{2}, 0\right)}$

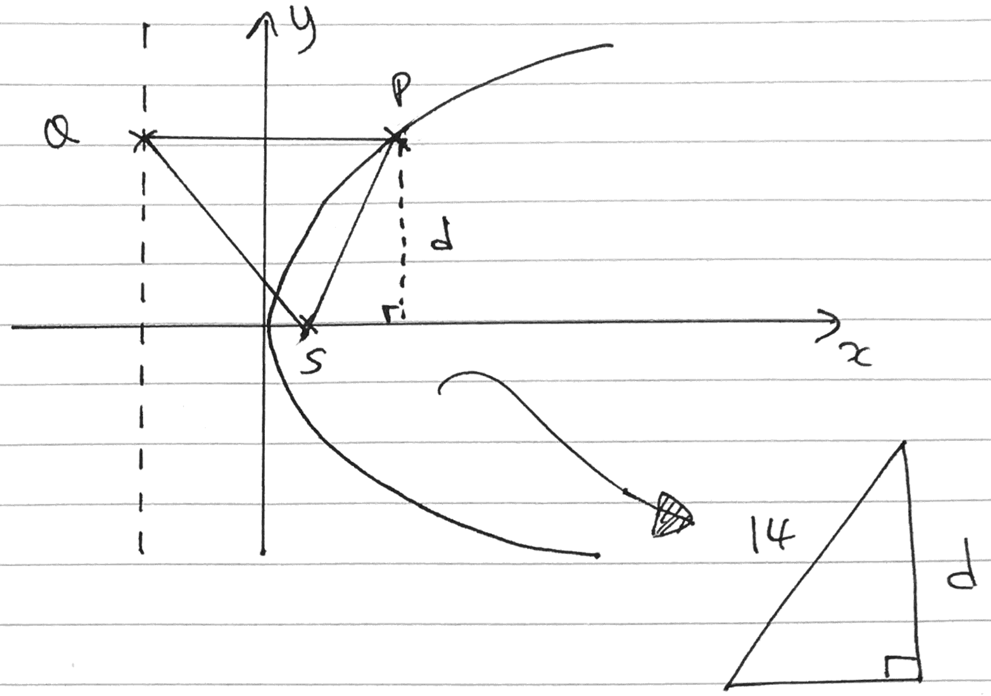
b) P will be equidistant from Q and S.

$\therefore \boxed{PS = 14}$

c) Directrix eqn: $x = -\frac{3}{2}$.

$\therefore$ x coordinate of $P = -\frac{3}{2} + 14$

$= \frac{25}{2}$ //



$\frac{25}{2} - \frac{3}{2}$

$= 11$

$\therefore d =$ y-coordinate of P

$= \sqrt{14^2 - 11^2} = 5\sqrt{3}$

so $\boxed{P\left(\frac{25}{2}, 5\sqrt{3}\right)}$

(Q4a) $A = \begin{pmatrix} 2p & 3q \\ 3p & 5q \end{pmatrix}$

$\det A = 2p(5q) - 3p(3q) = 10pq - 9pq$

$= pq$

$\therefore \boxed{A^{-1} = \frac{1}{pq}\begin{pmatrix} 5q & -3q \\ -3p & 2p \end{pmatrix}}$

b) $XA = B$

$XAA^{-1} = BA^{-1}$

$X = BA^{-1}$ //

$$\therefore X = \begin{pmatrix} p & q \\ 6p & 11q \\ 5p & 8q \end{pmatrix} \left(\frac{1}{pq}\right) \begin{pmatrix} 5q & -3q \\ -3p & 2p \end{pmatrix}$$

$$X = \frac{1}{pq} \begin{pmatrix} 5pq - 3pq & -3pq + 2pq \\ 30pq - 33pq & -18pq + 22pq \\ 25pq - 24pq & -15pq + 16pq \end{pmatrix}$$

$$X = \frac{1}{pq} \begin{pmatrix} 2pq & -pq \\ -3pq & 4pq \\ pq & pq \end{pmatrix}$$

$$X = \boxed{\begin{pmatrix} 2 & -1 \\ -3 & 4 \\ 1 & 1 \end{pmatrix}}$$

Q5a) $(z^2+9)(z^2+az+b) = z^4 + az^3 + bz^2 + 9z^2 + 9az + 9b$

compare coefficients with LHS of given eqn,

$\underline{z^3}$: $\boxed{a = -6}$ //

$\underline{z^2}$: $9 + b = 34$ $\therefore$ $\boxed{b = 25}$ //

b) $(z^2+9)(z^2-6z+25) = 0$

$z^2+9=0$: $z^2 = -9$
$z = \pm 3i$

$\therefore$ $\boxed{\begin{array}{l} z = 3i \\ z = -3i \end{array}}$

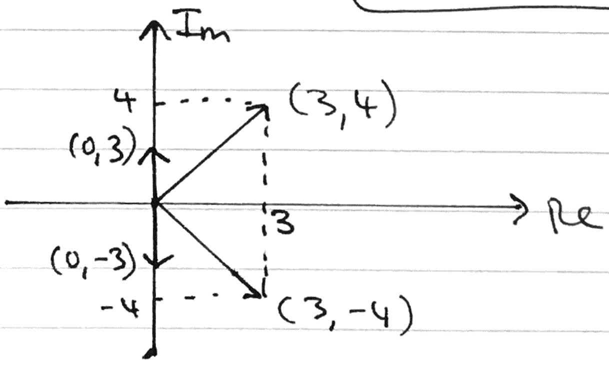
$z^2 - 6z + 25 = 0$: By Quadratic formula...

$\left.\begin{array}{l} a = 1 \\ b = -6 \\ c = 25 \end{array}\right\}$ $z = \dfrac{+6 \pm \sqrt{36 - 4(1)(25)}}{2}$

$z = 3 \pm \dfrac{8i}{2}$ //

so $\boxed{\begin{array}{l} z = 3+4i \\ z = 3-4i \end{array}}$

c)



Q6a) $f(x) = \dfrac{2x^3 + 6}{x^{\frac{1}{2}}} - 9$

$= 2x^{5/2} + 6x^{-\frac{1}{2}} - 9$ //

$\therefore f'(x) = 5x^{3/2} - 3x^{-3/2}$ //

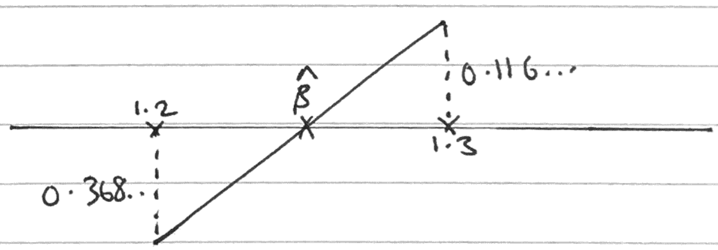
$f(0.45) = 0.21595...$

$f'(0.45) = -8.4287...$

so $\hat{\alpha} = 0.45 - \dfrac{0.21595...}{-8.4287...}$

$= \boxed{0.476}$ to 3 dp

b) $f(1.2) = -0.36789...$
$f(1.3) = 0.116141...$



$\Rightarrow \dfrac{0.116...}{0.368...} = \dfrac{1.3 - \hat{\beta}}{\hat{\beta} - 1.2}$

let $\dfrac{0.116...}{0.368...} = c$,

then $\dfrac{1.3 - \hat{\beta}}{\hat{\beta} - 1.2} = c$

$$c\hat{\beta} - 1.2c = 1.3 - \hat{\beta}$$

$$\hat{\beta}(c+1) = 1.3 + 1.2c$$

$$\therefore \hat{\beta} = \frac{1.3 + 1.2c}{c+1} = \boxed{1.276}$$

to 3 d.p.

Q7a) $5x^2 - 4x + 3 = 0$

$$x^2 - \frac{4}{5}x + \frac{3}{5} = 0$$

$$\therefore \alpha + \beta = \frac{4}{5}$$

$$\alpha\beta = \frac{3}{5} \quad //$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} \equiv \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} \equiv \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{\left(\frac{4}{5}\right)^2 - 2\left(\frac{3}{5}\right)}{\left(\frac{3}{5}\right)^2}$$

$$= \boxed{\frac{-14}{9}}$$

b) $\left(x - \left(\frac{3}{\alpha^2}\right)\right)\left(x - \left(\frac{3}{\beta^2}\right)\right) = 0$

expanding,

$$x^2 - x\left(\frac{3}{\beta^2} + \frac{3}{\alpha^2}\right) + \frac{9}{\alpha^2\beta^2} = 0$$

$$x^2 - 3x\left(\frac{1}{\beta^2} + \frac{1}{\alpha^2}\right) + \frac{9}{\left(\frac{3}{5}\right)^2} = 0$$

$$x^2 - 3x\left(\frac{-14}{9}\right) + 25 = 0$$

$$x^2 + \frac{14}{3}x + 25 = 0$$

$\times 3$: $\boxed{3x^2 + 14x + 75 = 0}$

Q8) $n=1$: $\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^1 = \text{RHS} = \begin{pmatrix} a^1 & 0 \\ \frac{a-b}{a-b} & b \end{pmatrix}$

LHS

$$= \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$\therefore$ true for $n=1$.

now assume the given relationship is true for $n=k$,

ie $$\left[\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ \frac{a^k-b^k}{a-b} & b^k \end{pmatrix} \quad // \right]$$

consider $n=k+1$,

$$\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}^k \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k & 0 \\ \frac{a^k-b^k}{a-b} & b^k \end{pmatrix} \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^k(a) & 0 \\ \frac{a(a^k-b^k)}{a-b} + b^k & b^k(b) \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \frac{a^{k+1}-b^k a}{a-b} + \frac{b^k(a-b)}{a-b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - b^k a + b^k a - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} a^{k+1} & 0 \\ \dfrac{a^{k+1} - b^{k+1}}{a - b} & b^{k+1} \end{pmatrix}$$

$\therefore$ true for $n = k+1$

So: true for $n = 1$
true for $n = k+1$ when true for $n = k$.

$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$

Q9 a) $\dfrac{z-4i}{z+3i} = i$

$zi + 3i^2 = z - 4i$

$zi - 3 = z - 4i$

$z(i-1) = 3 - 4i$

$z = \dfrac{3-4i}{-1+i}$

$\therefore z = \dfrac{(3-4i)(-1-i)}{(-1+i)(-1-i)} = \dfrac{-3-3i+4i+4i^2}{1+1}$

$z = \dfrac{-3-4+(4-3)i}{2}$

$z = \dfrac{-3-4}{2} + \dfrac{(4-3)}{2}i$

$z = -\left(\dfrac{4+3}{2}\right) + \left(\dfrac{4-3}{2}\right)i$

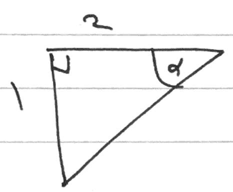
b i) $\underline{k=4}$: $z = -\dfrac{(4+3)}{2} + \dfrac{(4-3)}{2}i$

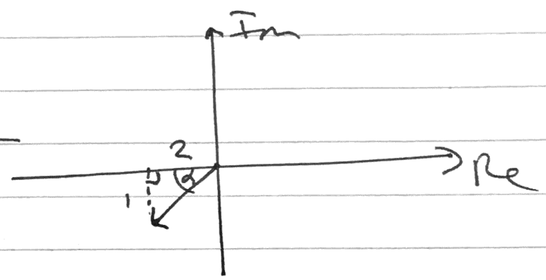
$$z = -\frac{7}{2} + \frac{1}{2}i$$

$$\therefore |z| = \sqrt{\left(-\tfrac{7}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = \boxed{\frac{5\sqrt{2}}{2}}$$

ii) $\underline{k=1}$: $z = -\dfrac{(1+3)}{2} + \dfrac{(1-3)}{2}i$

$$z = -2 - i$$





$\tan\alpha = \dfrac{1}{2}$

$\therefore \alpha = \arctan\dfrac{1}{2}$

clockwise from real axis is negative

so $\arg z = -\left[\pi - \arctan\tfrac{1}{2}\right] = \boxed{-2.678}$

Q10a) $xy = 144$ $\quad P\left(12p, \frac{12}{p}\right)$

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$ //

IMPLICIT DIFFERENTIATION

$\therefore \frac{dy}{dx} = -\frac{\frac{12}{p}}{12p} = -\frac{1}{p^2}$ //

so at normal to P, $m = p^2$.

$\left(p^2 \times \frac{-1}{p^2} = -1\right)$

$\Rightarrow y - \frac{12}{p} = p^2(x - 12p)$

$\Rightarrow y = p^2 x - 12p^3 + \frac{12}{p}$

b) $\underline{x=0}$: $y = \frac{12}{p} - 12p^3$

$$\therefore \boxed{R\left(0, \frac{12}{p} - 12p^3\right)}$$

$\underline{y=0}$: $p^2x + \frac{12}{p} - 12p^3 = 0$

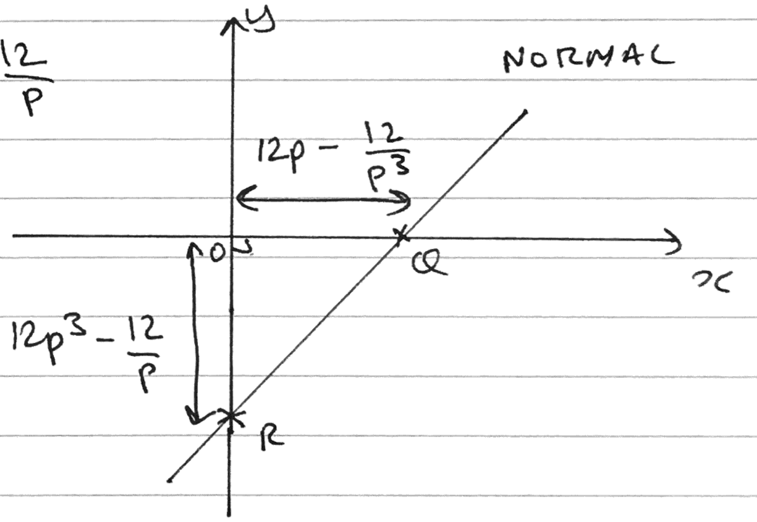
$$x = \frac{12p^3 - \frac{12}{p}}{p^2} = \frac{12p^3 - 12}{p^3}$$

$$= 12p - \frac{12}{p^3} \quad //$$

$$\text{so } \boxed{Q\left(12p - \frac{12}{p^3}, 0\right)}$$

c) length $OR = 12p^3 - \frac{12}{p}$
$\underline{\text{NOT}}$ $\frac{12}{p} - 12p^3$

as R is below the x-axis.



so Area $\Delta OQR = \frac{1}{2}\left(12p - \frac{12}{p^3}\right)\left(12p^3 - \frac{12}{p}\right)$

$$= \frac{12 \times 12}{2}\left(p - \frac{1}{p^3}\right)\left(p^3 - \frac{1}{p}\right) \quad //$$

$$\therefore 72\left(p^4 - 1 - 1 + \frac{1}{p^4}\right) = 512$$

$$p^4 + \frac{1}{p^4} - 2 = \frac{512}{72} = \frac{64}{9}$$

$\underline{\times 9p^4}$: $9p^8 - 82p^4 + 9 = 0$

$$(9p^4 - 1)(p^4 - 9) = 0$$

$9p^4 - 1 = 0$	$p^4 - 9 = 0$
$p^4 = \frac{1}{9}$	$p^4 = 9$
$\therefore p = \pm \frac{1}{\sqrt{3}}$ //	$\therefore p = \pm\sqrt{3}$ //

but note that we used $\left(12p^3 - \frac{12}{p}\right)$ as length OR as this was assumed to be positive since OR lies below the x-axis.

When using $p = \frac{1}{\sqrt{3}}$ or $p = -\sqrt{3}$ then the expression $\left(12p^3 - \frac{12}{p}\right)$ evaluates to < 0.

This would mean that R lies above the x-axis which is not true since we are told in the question that R is below the x-axis. So these values of p are invalid.

$\therefore \boxed{p = \sqrt{3}}$, $\boxed{p = -\frac{1}{\sqrt{3}}}$