

515 FP2

1. (a) Use algebra to find the set of values of x for which

$$x + 2 > \frac{12}{x + 3} \quad (6)$$

(b) Hence, or otherwise, find the set of values of x for which

$$x + 2 > \frac{12}{|x + 3|} \quad (1)$$

$$a) \quad (x+2)(x+3)^2 > \frac{12(x+3)^2}{(x+3)}$$

$$\Rightarrow (x+2)(x+3)^2 - 12(x+3) > 0$$

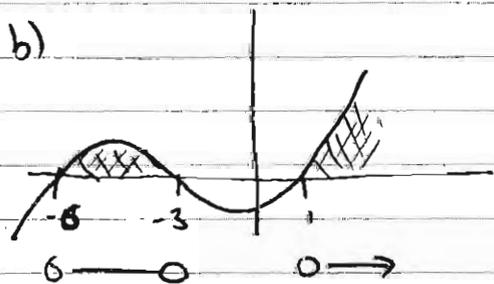
$$(x+3)[(x+2)(x+3) - 12] > 0$$

$$(x+3)(x^2 + 5x + 6 - 12) > 0$$

$$(x+3)(x+6)(x-1) > 0$$

-3
-6
1

b)



> 0

$$-6 < x < -3$$

or

$$x > 1$$

2. Further Mathematics · 2015 · May/June · FP2 zOE $-2 + (2\sqrt{3})i$

(a) Find the modulus and the argument of z .

(3)

Using de Moivre's theorem,

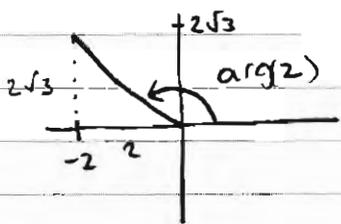
(b) find z^6 , simplifying your answer,

(2)

(c) find the values of w such that $w^4 = z^3$, giving your answers in the form $a + ib$ where $a, b \in \mathbb{R}$.

(4)

$$a) |z| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$



$$\begin{aligned} \arg(z) &= \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) \\ &= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

$$b) z = \cos\theta + i\sin\theta \Rightarrow z = 4 \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right]$$

$$z^n = (\cos n\theta + i\sin n\theta) \Rightarrow z^6 = 4^6 \left[\cos\left(\frac{12\pi}{3}\right) + i\sin\left(\frac{12\pi}{3}\right) \right]$$

$$\Rightarrow z^6 = 4^6 \times 1 = 4096$$

$$c) z^3 = 4^3 \left[\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) \right] = 64 [1] = 64$$

$$\therefore w^4 = 64 \left[\cos(0+2\pi k) + i\sin(0+2\pi k) \right]$$

$$\Rightarrow w = \sqrt[4]{64} \left[\cos\left(\frac{0+2\pi k}{4}\right) + i\sin\left(\frac{0+2\pi k}{4}\right) \right]$$

$$w = 2\sqrt{2} \left[\cos\left(\frac{1}{2}\pi k\right) + i\sin\left(\frac{1}{2}\pi k\right) \right]$$

$$k=0 \quad w_0 = 2\sqrt{2} (\cos 0 + i\sin 0) = 2\sqrt{2}$$

$$k=1 \quad w_1 = 2\sqrt{2} (\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})) = 2\sqrt{2}(i) = 2\sqrt{2}i$$

$$k=2 \quad w_2 = 2\sqrt{2} (\cos(\pi) + i\sin(\pi)) = 2\sqrt{2}(-1) = -2\sqrt{2}$$

$$k=3 \quad w_3 = 2\sqrt{2} (\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})) = 2\sqrt{2}(-i) = -2\sqrt{2}i$$

3. Find, in the form $y = f(x)$, the general solution of the differential equation
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$$\tan x \frac{dy}{dx} + y = 3 \cos 2x \tan x, \quad 0 < x < \frac{\pi}{2}$$

(6)

$$(\div \tan x) \quad \frac{dy}{dx} + \frac{1}{\tan x} y = 3 \cos 2x$$

$$IF = \int \frac{1}{\tan x} dx = \int \cot x dx = e^{\int \cot x dx} = e^{\ln |\sin x|} = \sin x$$

$$\Rightarrow \sin x \frac{dy}{dx} + \frac{\sin x}{\tan x} y = 3 \cos 2x \sin x$$

$$\Rightarrow \frac{d}{dx} (y \sin x) = 3 \cos 2x \sin x$$

$$\therefore y \sin x = 3 \int \cos 2x \sin x dx \quad \rightarrow \text{lots of methods to integrate}$$

$$y \sin x = 3 \int (2 \cos^2 x - 1) \sin x dx$$

$$= 3 \int 2 \sin x \cos^2 x - \sin x dx$$

$$= 6 \int \sin x \cos^2 x dx - 3 \int \sin x dx$$

$$\Rightarrow u = \cos x \quad + 3 \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{-du}{\sin x}$$

$$\therefore y \sin x = -2 \cos^3 x + 3 \cos x + c$$

$$= -6 \int u^2 du$$

$$= -\frac{6u^3}{3}$$

$$= -2 \cos^3 x$$

$$\therefore y = \frac{3 \cos x - 2 \cos^3 x + c}{\sin x}$$

2

4. (a) Show that
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$$r^2(r+1)^2 - (r-1)^2 r^2 \equiv 4r^3 \tag{3}$$

Given that $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \dots + n^3) = (1 + 2 + 3 + \dots + n)^2 \tag{4}$$

a) $r^2(r+1)^2 - (r-1)^2 r^2$
 $= r^2 [(r+1)^2 - (r-1)^2] = r^2 [r^2 + 2r + 1 - r^2 + 2r - 1] = r^2 \times 4r = 4r^3$

b) $1^3 + 2^3 + \dots + n^3 = \sum_1^n r^3$

$$\sum_1^n 4r^3 = \sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$$

$$\therefore \sum_1^n r^3 = \frac{1}{4} \sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$$

$$= \frac{1}{4} [(1^2 \times 2^2 - 0) + (2^2 \times 3^2 - 1^2 \times 2^2) + (3^2 \times 4^2 - 2^2 \times 3^2) + \dots + (n-1)^2 n^2 - (n-2)^2 (n-1)^2 + (n^2(n+1)^2 - (n-1)^2 n^2)]$$

$$= \frac{1}{4} n^2(n+1)^2 = \left[\frac{1}{2} n(n+1) \right]^2 = \left[\sum_1^n r \right]^2 = (1 + 2 + 3 + \dots + n)^2$$

$$\therefore 1^3 + 2^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

5. A transformation T from the z -plane to the w -plane is given by

$$w = \frac{z}{z + 3i}, \quad z \neq -3i$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C .

(a) (i) Show that C is a circle.

(ii) Find the centre and radius of C .

(8)

The region $|z| \leq 2$ in the z -plane is mapped by T onto the region R in the w -plane.

(b) Shade the region R on an Argand diagram.

(2)

$$a) \quad w(z+3i) = z \quad \Rightarrow \quad zw + 3wi = z$$

$$\Rightarrow 3wi = z - zw \quad \Rightarrow \quad z(1-w) = 3iw$$

$$\therefore z = \frac{3iw}{1-w}$$

$$|z| = 2 \quad \Rightarrow \quad \left| \frac{3iw}{1-w} \right| = 2 \quad \Rightarrow \quad |3iw| = 2|w-1|$$

$$\Rightarrow |i||3w| = 2|w-1|$$

$$w = u + iv \quad \Rightarrow \quad \Rightarrow |3w| = 2|w-1|$$

$$\Rightarrow |3u + 3iv| = 2|(u-1) + iv|$$

$$\Rightarrow |3u + 3iv| = |(2u-2) + 2iv|$$

$$\Rightarrow 9u^2 + 9v^2 = 4u^2 + 4 - 8u + 4v^2$$

$$\Rightarrow 5u^2 + 8u + 5v^2 = 4$$

$$\Rightarrow u^2 + \frac{8}{5}u + v^2 = \frac{4}{5}$$

$$\Rightarrow \left(u + \frac{4}{5}\right)^2 + v^2 = \frac{4}{5} + \frac{16}{25} = \frac{36}{25} = \left(\frac{6}{5}\right)^2$$

Circle, radius = $\frac{6}{5}$ ($-\frac{4}{5}, 0$) centre



test $z = 0 + 0i$ $w = \frac{0}{0+3i} = 0$

maps to
inside.

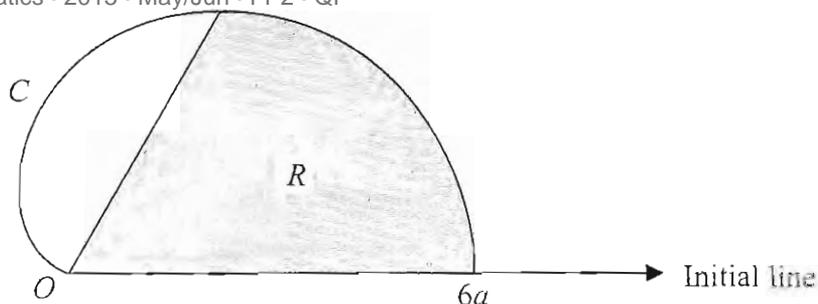


Figure 1

The curve C , shown in Figure 1, has polar equation

$$r = 3a(1 + \cos\theta), \quad 0 \leq \theta < \pi$$

The tangent to C at the point A is parallel to the initial line.

(a) Find the polar coordinates of A .

(6)

The finite region R , shown shaded in Figure 1, is bounded by the curve C , the initial line and the line OA .

(b) Use calculus to find the area of the shaded region R , giving your answer in the form $a^2(p\pi + q\sqrt{3})$, where p and q are rational numbers.

(5)

$$y = r \sin\theta = 3a(1 + \cos\theta)\sin\theta$$

$$y = 3a[(1 + \cos\theta)(\sin\theta)]$$

$$u = 1 + \cos\theta \quad v = \sin\theta$$

$$u' = -\sin\theta \quad v' = \cos\theta$$

$$\frac{dy}{d\theta} = 3a[-\sin^2\theta + \cos\theta + \cos^2\theta]$$

$$\frac{d}{d\theta} uv = vu' + uv'$$

$$= 3a[\cos^2\theta - 1 + \cos\theta + \cos^2\theta]$$

$$= 3a[2\cos^2\theta + \cos\theta - 1]$$

$$= 3a[(2\cos\theta - 1)(\cos\theta + 1)]$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2} \quad \cos\theta = -1$$

$$\theta = \frac{\pi}{3} \quad \theta = \pi \quad 0 \leq \theta < \pi$$

$$\therefore r = 3a(1 + \cos\frac{\pi}{3})$$

$$r = 3a \times \frac{3}{2} = \frac{9a}{2}$$

$$\therefore A\left(\frac{9a}{2}, \frac{\pi}{3}\right)$$

$$\begin{aligned}
 \text{b) Area} &= \frac{1}{2} \int_0^{\pi} r^2 d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{3}} 9a^2 (1 + \cos\theta)^2 d\theta \\
 &= \frac{9}{2} a^2 \int \cos^2\theta + 2\cos\theta + 1 d\theta \\
 &= \frac{9}{2} a^2 \int \left(\frac{1}{2} \cos 2\theta + \frac{1}{2} \right) + 2\cos\theta + 1 d\theta \\
 &= \frac{9}{4} a^2 \int \cos 2\theta + 4\cos\theta + 3 d\theta \\
 &= \frac{9}{4} a^2 \left[\frac{1}{2} \sin 2\theta + 4\sin\theta + 3\theta \right]_0^{\frac{\pi}{3}} \\
 &= \frac{9}{4} a^2 \left[\left(\frac{1}{2} \sin \frac{2\pi}{3} + 4\sin \frac{\pi}{3} + \pi \right) - (0 + 0 + 0) \right] \\
 &= \frac{9}{4} a^2 \left[\frac{9}{2} \sin \frac{\pi}{3} + \pi \right] = \frac{9}{4} a^2 \left[\frac{9\sqrt{3}}{4} + \pi \right] \\
 &= a^2 \left[\frac{81\sqrt{3}}{16} + \frac{9\pi}{4} \right]
 \end{aligned}$$

$$c) \text{ Further Mathematics } \cdot 2015 \cdot \text{ May/June } \cdot \text{ P2 } \cdot \text{ QP} \quad \tan x = \tan \frac{\pi}{3} = \sqrt{3}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\cos \frac{\pi}{3}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$y = \tan^2 x = (\sqrt{3})^2 = 3$$

$$\frac{dy}{dx} = 2 \tan x \sec^2 x = 2(\sqrt{3})(2)^2 = 8\sqrt{3}$$

$$\frac{d^2y}{dx^2} = 6 \sec^4 x - 4 \sec^2 x = 6(2)^4 - 4(2)^2 = 80$$

$$\frac{d^3y}{dx^3} = 8 \sec^2 x \tan x (3 \sec^2 x - 1) = 8(2)^2(\sqrt{3})(3(2)^2 - 1) = 352\sqrt{3}$$

$$\therefore \tan^2 x \approx 3 + 8\sqrt{3}\left(x - \frac{\pi}{3}\right) + 40\left(x - \frac{\pi}{3}\right)^2 + \frac{352}{6}\sqrt{3}\left(x - \frac{\pi}{3}\right)^3$$

8. (a) Show that the transformation $x = e^u$ transforms the differential equation

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$$x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 16y = 2 \ln x, \quad x > 0 \quad (\text{I})$$

into the differential equation

$$\frac{d^2 y}{du^2} - 8 \frac{dy}{du} + 16y = 2u \quad (\text{II})$$

(6)

(b) Find the general solution of the differential equation (II), expressing y as a function of u .

(7)

(c) Hence obtain the general solution of the differential equation (I).

(1)

$$a) x = e^u \Rightarrow \frac{dx}{du} = e^u = x$$

$$\ln x = u \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x} \frac{dy}{du}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{1}{x} \frac{dy}{du} \right]$$

$$u = \frac{1}{x} \quad v = \frac{dy}{du}$$

$$u' = -\frac{1}{x^2} \quad v' = \frac{d}{dx} \left(\frac{dy}{du} \right)$$

$$= \frac{1}{x^2} \frac{d^2y}{du^2} - \frac{1}{x^2} \frac{dy}{du}$$

$$= \frac{du}{dx} \times \frac{d}{du} \left(\frac{dy}{du} \right)$$

$$= \frac{1}{x} \times \frac{d^2y}{du^2}$$

$$x^2 \frac{d^2y}{dx^2} - 7x \frac{dy}{dx} + 16y = 2 \ln x$$

$$x^2 \left(\frac{1}{x^2} \frac{d^2y}{du^2} - \frac{1}{x^2} \frac{dy}{du} \right) - 7x \left(\frac{1}{x} \frac{dy}{du} \right) + 16y = 2u$$

$$\Rightarrow \frac{d^2y}{du^2} - \frac{dy}{du} - 7 \frac{dy}{du} + 16y = 2u \quad \therefore \frac{d^2y}{du^2} - 8 \frac{dy}{du} + 16y = 2u$$

b) (PI) $y = au + b$
 $y' = a$
 $y'' = 0$

$$y'' - 8y' + 16y = 2u$$

$$0 - 8a + 16au + 16b = 2u$$

$$16au = 2u \quad \therefore a = \frac{1}{8}$$

$$y_{PI} = \frac{1}{8}u + \frac{1}{16}$$

$$16b - 8a = 0$$

$$16b = 1 \quad \therefore b = \frac{1}{16}$$

(CF) $y = Ae^{mu}$
 $y' = Ame^{mu}$
 $y'' = Am^2e^{mu}$

$$y'' - 8y' + 16y = 0$$

$$Am^2e^{mu} - 8Ame^{mu} + 16Ae^{mu} = 0$$

$$Ae^{mu}(m^2 - 8m + 16) = 0$$

$$\neq 0 \quad (m - 4)^2 \quad \therefore m = 4$$

$$\therefore y_{CF} = (A + Bu)e^{4u}$$

$$\therefore y_{GS} = (A + Bu)e^{4u} + \frac{1}{8}u + \frac{1}{16}$$

c) $u = \ln x$ $y = (A + B \ln x)e^{4 \ln x} + \frac{1}{8} \ln x + \frac{1}{16}$

$$y = (A + B \ln x)(e^{\ln x})^4 + \frac{1}{8} \ln x + \frac{1}{16}$$

$$y = (A + B \ln x)x^4 + \frac{1}{8} \ln x + \frac{1}{16}$$

Z