

FP2 June 2017 (MA)

$$\text{Q1a) LHS} = \frac{(r+1)^2}{r^2(r+1)^2} - \frac{r^2}{r^2(r+1)^2} = \frac{r^2 + 2r + 1 - r^2}{r^2(r+1)^2}$$

$$= \boxed{\frac{2r+1}{r^2(r+1)^2}}$$

$$\text{b) } \sum_{r=1}^n \frac{1}{r^2} - \frac{1}{(r+1)^2} : \underline{n=1} : 1 - \frac{1}{4}$$

$$\underline{n=2} : \frac{1}{4} - \frac{1}{9}$$

$$\underline{n=n-1} : \frac{1}{(n-1)^2} - \frac{1}{n^2}$$

$$\underline{n=n} : \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$\therefore \sum_{r=1}^n \frac{1}{r^2} - \frac{1}{(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2}{(n+1)^2} - \frac{1}{(n+1)^2} = \frac{n^2 + 2n + 1 - 1}{(n+1)^2}$$

$$= \frac{n^2 + 2n}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

$$\text{c) } \sum_{r=n}^{3n} \dots = \sum_{r=1}^{3n} \dots - \sum_{r=1}^{n-1} \dots$$

$$\Rightarrow \frac{3n(3n+2)}{(3n+1)^2} - \frac{(n-1)(n-1+2)}{(n-1+1)^2} = \sum_{r=n}^{3n} \frac{2r+1}{r^2(r+1)^2} //$$

$$\therefore \sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2} = \frac{9n^2+6n}{(3n+1)^2} - \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{(9n^2+6n)(n^2) - (n^2-1)(3n+1)^2}{n^2(3n+1)^2}$$

$$= \frac{9n^4 + 6n^3 - [n^2-1][9n^2+6n+1]}{n^2(3n+1)^2}$$

$$= \frac{9n^4 + 6n^3 - [9n^4 + 6n^3 + n^2 - 9n^2 - 6n - 1]}{n^2(3n+1)^2}$$

$$= \frac{9n^4 + 6n^3 - 9n^4 - 6n^3 + 8n^2 + 6n + 1}{n^2(3n+1)^2}$$

$$= \frac{8n^2 + 6n + 1}{n^2(3n+1)^2} = \sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2}$$

$$\text{So } \sum_{r=1}^{3n} \frac{6r+3}{r^2(r+1)^2} = 3 \times \left[\frac{8n^2 + 6n + 1}{n^2(3n+1)^2} \right]$$

$$= \boxed{\frac{24n^2 + 18n + 3}{n^2(3n+1)^2}}$$

Q2) multiply both sides by $(x+2)^2$:

$$\frac{(x-2)(x+2)}{2} \leq \frac{12}{x}(x+2)$$

multiply by $2x^2$ next:

$$x^2(x-2)(x+2) \leq 12(2)x(x+2)$$

$$x^2(x+2)(x-2) \leq 24x(x+2)$$

$$x(x+2)[x(x-2) - 24] \leq 0$$

$$x(x+2)[x^2 - 2x - 24] \leq 0$$

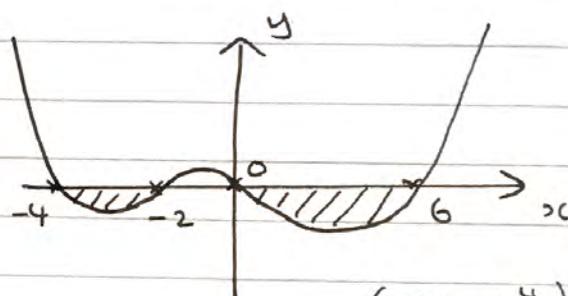
$$x(x+2)(x-6)(x+4) \leq 0$$

critical values: $x=0$

$$x=-2$$

$$x=6$$

$$x=-4$$



so the region required is given by:

$$\boxed{0 < x \leq 6} \quad , \quad \boxed{-4 \leq x < -2}$$

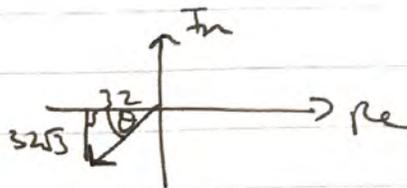
REMEMBER: in the given ~~equation~~ inequality, (x) and $(x+2)$ are on the denominators of the fractions. So $x \neq 0$ and $x \neq -2$.
(These equations are not defined for these values of x)

$$3) \quad z^3 = -32 - 32i\sqrt{3}$$

$$|z^3| = \sqrt{(-32)^2 + (-32\sqrt{3})^2} = 64 //$$

$$\arg z^3 = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3} //$$

$$\left[\tan \theta = \sqrt{3} : \theta = \frac{\pi}{3}\right]$$



$$\therefore z^3 = 64 \left[e^{\left(-\frac{2\pi}{3}\right)i} \right] = 64 \left[e^{\left(-\frac{2\pi}{3} + 2\pi k\right)i} \right]$$

$$z = 64^{\frac{1}{3}} \left[e^{\left(-\frac{2\pi}{3} + \frac{6\pi k}{3}\right)i} \right]^{\frac{1}{3}}$$

$$z = 4 e^{\frac{(6\pi k - 2\pi)}{9}i} //$$

$$\underline{k=0}: z = 4 e^{-\frac{2\pi i}{9}} //$$

$$\underline{k=1}: z = 4 e^{\frac{4\pi i}{9}} //$$

$$\underline{k=-1}: z = 4 e^{-\frac{8\pi i}{9}} //$$

$$4a) y = \ln((1-2x)^{-1})$$

$$\frac{dy}{dx} = \frac{- (1-2x)^{-2} (-2)}{(1-2x)^{-1}} = 2(1-2x)^{-1} //$$

$$\frac{d^2y}{dx^2} = -2(1-2x)^{-2} (-2) = 4(1-2x)^{-2} //$$

$$\frac{d^3y}{dx^3} = -8(1-2x)^{-3} (-2) = 16(1-2x)^{-3} //$$

$$b) \left. \begin{array}{l} f(0) = \ln(1) = 0 \\ f'(0) = 2 \\ f''(0) = 4(1) = 4 \\ f'''(0) = 16 \end{array} \right\} \begin{array}{l} y \approx 0 + 2x + \frac{4x^2}{2} + \frac{16x^3}{6} \\ \boxed{y \approx 2x + 2x^2 + \frac{8x^3}{3}} \end{array}$$

$$c) \frac{1}{1-2x} = \frac{3}{2} \rightarrow 1-2x = \frac{2}{3} \therefore 2x = \frac{1}{3}$$

$$\text{so } x = \frac{1}{6} //$$

$$\text{using } x = \frac{1}{6}: \ln\left(\frac{1}{1-2x}\right) \rightarrow \ln\left(\frac{3}{2}\right)$$

$$\ln\left(\frac{3}{2}\right) \approx 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right)^2 + \frac{8}{3}\left(\frac{1}{6}\right)^3$$

$$\approx \boxed{0.401}$$

5a) AUX: $\lambda^2 - 2\lambda = 0$
 $\lambda(\lambda - 2) = 0$
 $\lambda = 0, \lambda = 2$

C.F: $y = A + Be^{2x}$

P.I: let $y = \lambda \sin 3x + \varphi \cos 3x$
then $\dot{y} = 3\lambda \cos 3x - 3\varphi \sin 3x$
and $\ddot{y} = -9\lambda \sin 3x - 9\varphi \cos 3x$

subbing back: $-9\lambda \sin 3x - 9\varphi \cos 3x - 6\lambda \cos 3x + 6\varphi \sin 3x = 26 \sin 3x$

comparing coefficients: $-9\lambda + 6\varphi = 26 \sim \textcircled{1}$
 $-9\varphi - 6\lambda = 0 \sim \textcircled{2}$

from $\textcircled{2}$: $\lambda = -\frac{3\varphi}{2}$

$\hookrightarrow \textcircled{1}$: $\frac{27}{2}\varphi + 6\varphi = 26$
 $\therefore \varphi = \frac{4}{3} //$
and $\lambda = -2 //$

General solution: $y = A + Be^{2x} - 2\sin 3x + \frac{4}{3}\cos 3x$

b) $y=0, x=0$: $0 = A + B + \frac{4}{3} \therefore A = -\frac{4}{3} - B //$

$\frac{dy}{dx} = 2Be^{2x} - 6\cos 3x - 4\sin 3x$

$\frac{dy}{dx}=0, x=0$: $0 = 2B - 6 \therefore B = 3 //$
and so $A = -\frac{4}{3} - 3 = -\frac{13}{3} //$

$\therefore y = -\frac{13}{3} + 3e^{2x} - 2\sin 3x + \frac{4}{3}\cos 3x$

$$6) \text{ Area} = \frac{1}{2} \int_0^{2\pi} (r^2) d\theta = \frac{1}{2} \int_0^{2\pi} (6 + a \sin \theta)^2 d\theta = \frac{97\pi}{2}$$

$$= \frac{1}{2} \int_0^{2\pi} [36 + 12a \sin \theta + a^2 \sin^2 \theta] d\theta$$

$$\begin{aligned} \cos 2\theta &= 1 - 2\sin^2 \theta \\ \sin^2 \theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

$$= \frac{1}{2} \int_0^{2\pi} \left[36 + 12a \sin \theta + a^2 \left[\frac{1}{2} - \frac{1}{2} \cos 2\theta \right] \right] d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[36 + \frac{a^2}{2} + 12a \sin \theta - \frac{1}{2} a^2 \cos 2\theta \right] d\theta$$

$$= \frac{1}{2} \left[36\theta + \frac{a^2}{2}\theta - 12a \cos \theta - \frac{1}{4} a^2 \sin 2\theta \right]_0^{2\pi}$$

$$= \frac{1}{2} [72\pi + a^2\pi - 12a] - \frac{1}{2} [-12a] = \frac{97\pi}{2}$$

$$\Rightarrow \frac{\pi(72 + a^2) - 12a + 12a}{2} = \frac{97\pi}{2}$$

$$\Rightarrow \pi(72 + a^2) = 97\pi$$

$$\Rightarrow a^2 = 97 - 72 = 25 \\ \therefore a = \sqrt{25} = \boxed{5}$$

$$(a > 0)$$

$$7a) \cos x \frac{dy}{dx} + y \sin x = 2 \cos^3 x \sin x + 1$$

$$\textcircled{-\cos x} : \frac{dy}{dx} + y \tan x = 2 \cos^2 x \sin x + \sec x.$$

$$I = e^{\int \tan x \, dx} = e^{\ln |\sec x|} = \underline{\underline{\sec x}}$$

$$\textcircled{\times \sec x} : \sec x \frac{dy}{dx} + y \sec x \tan x = 2 \cos x \sin x + \sec^2 x$$

$$\Rightarrow \frac{d}{dx} (y \sec x) = 2 \cos x \sin x + \sec^2 x$$

$$\Rightarrow y \sec x = \int (\sin 2x) \, dx + \int (\sec^2 x) \, dx$$

$$\Rightarrow y \sec x = -\frac{1}{2} \cos 2x + \tan x + c$$

$$\Rightarrow y = \cos x \left(\tan x - \frac{1}{2} \cos 2x + c \right)$$

$$b) 5\sqrt{2} = \cos \frac{\pi}{4} \left(\tan \frac{\pi}{4} - \frac{1}{2} \cos \frac{\pi}{2} + c \right)$$

$$5\sqrt{2} = \frac{\sqrt{2}}{2} (1 + c)$$

$$10 = 1 + c \quad \therefore c = 9 //$$

$$y = \cos x \left(\tan x - \frac{1}{2} \cos 2x + 9 \right)$$

$$\underline{x = \frac{\pi}{6}} : y = \frac{\sqrt{3}}{2} \left(\frac{\sqrt{3}}{3} - \frac{1}{4} + 9 \right) = \frac{3}{6} + \frac{\sqrt{3}}{2} \left(\frac{35}{4} \right)$$

$$\boxed{y = \frac{1}{2} + \frac{35\sqrt{3}}{8}}$$

8a) make z the subject :

$$(1+iz)w = z+3i$$

$$w + wz_i - z = 3i$$

$$z(w_i - 1) = 3i - w$$

$$z = \frac{3i - w}{w_i - 1}$$

$$|z| = \frac{|3i - w|}{|w_i - 1|} = 1$$

$$\therefore |w_i - 1| = |3i - w|$$

$$|(u+iv)i - 1| = |3i - u - iv|$$

$$|ui - 1 - v| = |-u + i(3-v)|$$

$$|(-1-v) + i(u)| = |-u + i(3-v)|$$

squaring : $(-1-v)^2 + u^2 = u^2 + (3-v)^2$

$$v^2 + 2v + 1 = v^2 + 9 - 6v$$

$$8v = 8$$

$$\boxed{v=1}$$

$$b) |z - (a+bi)| = c \text{ in } z\text{-plane.}$$

$$|w| = 5 \text{ in } w\text{-plane.}$$

make w the subject now:

$$w = \frac{z+3i}{1+zi}$$

$$|w| = \frac{|z+3i|}{|1+zi|} = 5 //$$

$$\therefore |z+3i| = 5|1+zi|$$

$$|(x+iy+3i)| = 5|1+(x+iy)i|$$

$$|x+i(y+3)| = 5|1+xi-y|$$

$$|x+i(y+3)| = 5|(1-y)+i(x)|$$

Squaring: $x^2 + (y+3)^2 = 25[(1-y)^2 + x^2]$

$$x^2 + y^2 + 6y + 9 = 25[1 - 2y + x^2 + y^2]$$

$$x^2 + y^2 + 6y + 9 = 25 - 50y + 25x^2 + 25y^2$$

$$24x^2 + 24y^2 - 56y + 16 = 0$$

$$\div 24: x^2 + y^2 - \frac{7}{3}y + \frac{2}{3} = 0$$

centre: $(\frac{7}{6}, 0)$
 $r = \frac{5}{6}$

$$(x - \frac{7}{6})^2 + y^2 = \frac{49}{36} - \frac{2}{3}$$

$$(x - \frac{7}{6})^2 + y^2 = \frac{25}{36} //$$

$a = \frac{7}{6}$
$b = 0$
$c = \frac{5}{6}$