

FP2 June 2018 (MA)

$$Q(a) \quad \frac{1}{(r+3)(r+4)} = \frac{A}{r+3} + \frac{B}{r+4}$$

$$1 = A(r+4) + B(r+3)$$

$$\underline{r = -4} : 1 = -B \quad \therefore B = -1 //$$

$$\underline{r = 0} : 1 = 4A + 3(-1) \quad \therefore A = 1 //$$

$$\therefore \frac{1}{(r+3)(r+4)} = \boxed{\frac{1}{r+3} - \frac{1}{r+4}}$$

$$b) \quad \sum_1^n \frac{1}{(r+3)(r+4)} = \sum \frac{1}{r+3} - \frac{1}{r+4} //$$

$$\underline{n=1} : \frac{1}{4} - \frac{1}{5}$$

$$\underline{n=2} : \frac{1}{5} - \frac{1}{6}$$

...

$$\underline{n=n-1} : \frac{1}{n+2} - \frac{1}{n+3}$$

$$\underline{n=n} : \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{1}{4} - \frac{1}{n+4}$$

$$= \frac{(n+4) - (4)}{4(n+4)}$$

$$= \boxed{\frac{n}{4(n+4)}}$$

$$c) \sum_{r=15}^{30} \frac{1}{(r+3)(r+4)} = \sum_{r=1}^{30} \frac{1}{(r+3)(r+4)} - \sum_{r=1}^{14} \frac{1}{(r+3)(r+4)}$$

$$\therefore \sum_{15}^{30} \frac{1}{(r+3)(r+4)} = \frac{30}{4(30+4)} - \frac{14}{4(14+4)}$$

$$= \boxed{\frac{4}{153}}$$

Q2) We are given information about the  $z$ -plane (points lie on the real-plane) so make  $z$  the subject:

$$w = \frac{1-iz}{z}$$

$$zw = 1-iz$$

$$z(w+i) = 1$$

$$z = \frac{1}{w+i}$$

$$\therefore x+iy = \frac{1}{u+iv+i} = \frac{1}{u+i(v+1)}$$

$$x+iy = \frac{u-i(v+1)}{(u+i(v+1))(u-i(v+1))} = \frac{u-i(v+1)}{u^2+(v+1)^2}$$

$$\therefore x+iy = \frac{u}{u^2+(v+1)^2} - \frac{(v+1)i}{u^2+(v+1)^2}$$

points lie on the real axis in  $z$ -plane so  $y=0$ .

$$y = \frac{-(v+1)}{u^2 + (v+1)^2} = 0$$

$$-v - 1 = 0$$

$$\boxed{v = -1}$$

alt :  $w = \frac{1-iz}{z}$

$$u+iv = \frac{1-i(x+iy)}{x+iy}$$

$y=0$  :  $u+iv = \frac{1-xi}{x}$

$$u+iv = \frac{1}{x} - i$$

compare imaginary terms :  $\boxed{v = -1}$

$$\begin{aligned}
 \text{(Q3ai)} \quad \sin \frac{\pi}{12} &= \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} \\
 &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\
 &= \boxed{\frac{1}{4} (\sqrt{6} - \sqrt{2})}
 \end{aligned}$$

$$\begin{aligned}
 \text{ii)} \quad \cos \frac{\pi}{12} &= \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \\
 &= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \\
 &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{1}{4} (\sqrt{6} + \sqrt{2})}
 \end{aligned}$$

$$\text{b)} \quad z^4 = 4 \left( \cos \left( \frac{\pi}{3} + 2k\pi \right) + i \sin \left( \frac{\pi}{3} + 2k\pi \right) \right)$$

$$z = 4^{\frac{1}{4}} \left( \cos \left( \frac{6k\pi + \pi}{3} \right) + i \sin \left( \frac{\pi + 6k\pi}{3} \right) \right)^{\frac{1}{4}}$$

$$z = \sqrt{2} \left( \cos \left( \frac{6k\pi + \pi}{12} \right) + i \sin \left( \frac{6k\pi + \pi}{12} \right) \right)$$

$$\underline{k=0} : z = \sqrt{2} \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$\boxed{z = \frac{\sqrt{2}}{4} (\sqrt{6} + \sqrt{2}) + i \frac{\sqrt{2}}{4} (\sqrt{6} - \sqrt{2})}$$

$$\underline{k=1} : z = \sqrt{2} \left( \cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12} \right)$$

$$\boxed{z = \frac{\sqrt{2}}{4} (-\sqrt{6} + \sqrt{2}) + i \frac{\sqrt{2}}{4} (\sqrt{6} + \sqrt{2})}$$

$$u=2: z = \sqrt{2} \left( \cos \frac{13\pi}{12} + i \sin \frac{13\pi}{12} \right)$$

$$z = -\frac{\sqrt{2}}{4} (\sqrt{6} + \sqrt{2}) + \frac{i\sqrt{2}}{4} (-\sqrt{6} + \sqrt{2})$$

$$u=3: z = \sqrt{2} \left( \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12} \right)$$

$$z = \frac{\sqrt{2}}{4} (\sqrt{6} - \sqrt{2}) + \frac{i\sqrt{2}}{4} (-\sqrt{6} - \sqrt{2})$$

$$z = \frac{\sqrt{2}}{4} (\sqrt{6} - \sqrt{2}) - \frac{i\sqrt{2}}{4} (\sqrt{6} + \sqrt{2})$$

Q4)  $4x = x^2 - 2$

$$x^2 - 4x - 2 = 0$$

By Quadratic formula:

$$x = 2 \pm \sqrt{6} //$$

$x > 0$  for any intersection so  $x = 2 + \sqrt{6} //$

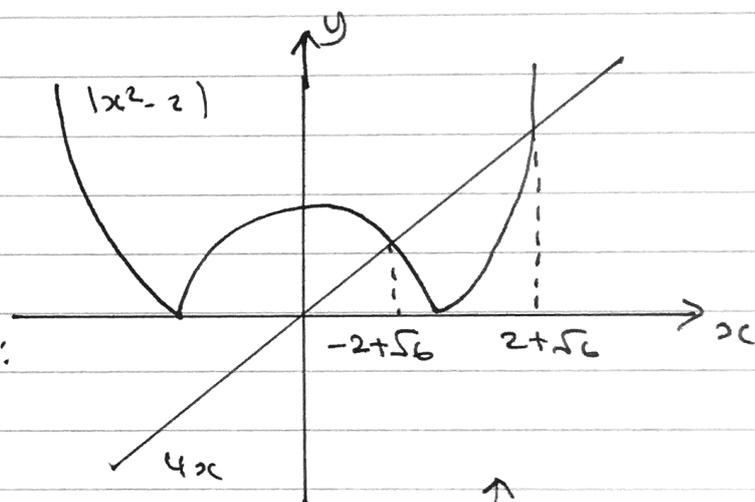
$$4x = -(x^2 - 2)$$

$$x^2 + 4x - 2 = 0$$

By Quadratic formula:

$$x = -2 \pm \sqrt{6} //$$

$x > 0$  for any intersection so  $x = -2 + \sqrt{6} //$



region required is where the curve is above the line.

where:

$$\boxed{\begin{array}{l} x < -2 + \sqrt{6} \\ x > 2 + \sqrt{6} \end{array}}$$

(Q5a)  $\frac{d}{dx}(y y'' + 3x y' - 3y^2) = 0$

$$\Rightarrow y' y'' + y y''' + 3y' + 3x y'' - 6y y' = 0$$

$$\Rightarrow y''' = \frac{6y y' - 3x y'' - 3y' - y' y''}{y} //$$

using original eqn:

$$\underline{x=0, y=2, y'=1} : 2y'' + 3(0) - 3(2)^2 = 0$$

$$y'' = \frac{12}{2} = 6 //$$

now finding  $y'''$ :  $y''' = \frac{6(2)(1) - 0 - 3(1) - 1(6)}{2}$

$$= \boxed{\frac{3}{2}} //$$

b)  $y=2, y'=1, y''=6, y'''=\frac{3}{2}$

$$\therefore y \approx 2 + x + \frac{6x^2}{2} + \frac{\frac{3}{2}x^3}{6}$$

$$\Rightarrow \boxed{y \approx 2 + x + 3x^2 + \frac{1}{4}x^3}$$

● (Q6a)  $6 \frac{d^2y}{dx^2} + 5 \frac{dy}{dx} - 6y = x - 6x^2$

AUX:  $6\lambda^2 + 5\lambda - 6 = 0$

$$(3\lambda - 2)(2\lambda + 3) = 0$$

$$\therefore \lambda = \frac{2}{3}, \lambda = -\frac{3}{2}$$

● C.F:  $y = Ae^{\frac{2}{3}x} + Be^{-\frac{3}{2}x}$

P.I: let  $y = dx^2 + \mu x + \psi$

$$\dot{y} = 2dx + \mu$$

$$\ddot{y} = 2d$$

● Substituting:  $6(2d) + 5(2dx + \mu) - 6(dx^2 + \mu x + \psi) = x - 6x^2$

$$= x - 6x^2$$

$$\Rightarrow 12d + 10dx + 5\mu - 6dx^2 - 6\mu x - 6\psi = x - 6x^2$$

comparing coefficients:  $x^2$ :  $-6d = -6$

$$\therefore d = 1 //$$

$x$ :  $10d - 6\mu = 1 \quad \therefore \mu = \frac{10-1}{6} = \frac{3}{2} //$

constants:  $12d + 5\mu - 6\psi = 0 \quad \therefore \psi = \frac{13}{4} //$

So...

$$\text{General solution: } y = Ae^{\frac{2}{3}x} + Be^{-\frac{3}{2}x} + x^2 + \frac{3}{2}x + \frac{13}{4}$$

b)  $y=0, x=0$  :  $0 = A + B + \frac{13}{4}$

$$\therefore A + B = -\frac{13}{4} \quad \text{--- (1)}$$

$\frac{dy}{dx} = \frac{3}{2}$ ,  $x=0$  :  $\frac{dy}{dx} = \frac{2}{3}Ae^{\frac{2}{3}x} - \frac{3}{2}Be^{-\frac{3}{2}x} + 2x + \frac{3}{2}$

$$\frac{3}{2} = \frac{2}{3}A - \frac{3}{2}B + \frac{3}{2}$$

$\times 6$  :  $4A - 9B = 0$

$$A = \frac{9B}{4}$$

sub A into (1) :  $\frac{9B}{4} + B = -\frac{13}{4}$

$$B\left(1 + \frac{9}{4}\right) = -\frac{13}{4}$$

$$\therefore B = \frac{-\frac{13}{4}}{1 + \frac{9}{4}} = -1$$

and  $A = \frac{9}{4} \times -1 = -\frac{9}{4}$

so particular solution is...

$$y = -\frac{9}{4}e^{\frac{2}{3}x} - e^{-\frac{3}{2}x} + x^2 + \frac{3}{2}x + \frac{13}{4}$$

● (Q7a) tangent parallel to initial line  $\rightarrow \frac{dy}{d\theta} = 0$ .

$$r = 2 + \sqrt{3} \cos \theta$$

$$r \sin \theta = 2 \sin \theta + \sqrt{3} \sin \theta \cos \theta$$

$$y = 2 \sin \theta + \sqrt{3} \sin \theta \cos \theta = \sin \theta (2 + \sqrt{3} \cos \theta)$$

$$\frac{dy}{d\theta} = \cos \theta (2 + \sqrt{3} \cos \theta) + \sin \theta (-\sqrt{3} \sin \theta) = 0$$

$$\Rightarrow 2 \cos \theta + \sqrt{3} \cos^2 \theta - \sqrt{3} \sin^2 \theta = 0$$

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\Rightarrow 2 \cos \theta + \sqrt{3} \cos^2 \theta + \sqrt{3} \cos^2 \theta - \sqrt{3} = 0$$

$$\Rightarrow 2\sqrt{3} \cos^2 \theta + 2 \cos \theta - \sqrt{3} = 0$$

By Quadratic formula:  $\cos \theta = \frac{\sqrt{21} \pm \sqrt{3}}{6}$

$$\cos \theta \leq 1 \quad \therefore \cos \theta = \frac{\sqrt{21} - \sqrt{3}}{6}$$

$$\text{distance } OP = r = 2 + \sqrt{3} \left( \frac{\sqrt{21} - \sqrt{3}}{6} \right)$$

$$= 2 + \frac{3\sqrt{7} - 3}{6}$$

$$= 2 + \frac{\sqrt{7}}{2} - \frac{1}{2}$$

$$= \frac{3}{2} + \frac{\sqrt{7}}{2} = \boxed{\frac{1}{2} (3 + \sqrt{7})}$$

$$b) \text{ Area} = \frac{1}{2} \int_0^{2\pi} r^2 d\theta = \frac{1}{2} \int_0^{2\pi} (2 + \sqrt{3} \cos \theta)^2 d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (4 + 4\sqrt{3} \cos \theta + 3 \cos^2 \theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (4 + 4\sqrt{3} \cos \theta + \frac{3}{2} + \frac{3}{2} \cos 2\theta) d\theta$$

$$\begin{aligned} \cos 2\theta &= 2\cos^2 \theta - 1 \\ \therefore \cos^2 \theta &= \frac{1 + \cos 2\theta}{2} \end{aligned}$$

$$= \frac{1}{2} \int_0^{2\pi} (4 + 4\sqrt{3} \cos \theta + \frac{3}{2} + \frac{3}{2} \cos 2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (\frac{11}{2} + 4\sqrt{3} \cos \theta + \frac{3}{2} \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left[ \frac{11\theta}{2} + 4\sqrt{3} \sin \theta + \frac{3}{4} \sin 2\theta \right]_0^{2\pi}$$

$$= \frac{1}{2} [11\pi + 0 + 0] - \frac{1}{2} [0]$$

$$= \boxed{\frac{11\pi}{2}}$$

● Q8a)

$$\int (2x^5 e^{-x^2}) dx$$

$$t = x^2$$

$$\frac{dt}{dx} = 2x$$

$$\Downarrow$$

$$dx = \frac{1}{2x} dt$$

$$\int \left( 2x^5 e^{-t} \times \frac{1}{2x} \right) dt$$

$$= \int (x^4 e^{-t}) dt = \int (t^2 e^{-t}) dt //$$

By parts:  $\frac{dv}{dt} = e^{-t} \rightarrow v = -e^{-t}$

$$u = t^2 \rightarrow u' = 2t$$

$$= [-t^2 e^{-t}] + 2 \int (t e^{-t}) dt$$

By parts (again)

$$\frac{dv}{dt} = e^{-t} \rightarrow v = -e^{-t}$$

$$u = t \rightarrow u' = 1$$

$$\therefore \int (t e^{-t}) dt = [-t e^{-t}] + \int (e^{-t}) dt$$

$$\text{so } \int (te^{-t}) dt = -te^{-t} - e^{-t}$$

$$\therefore \int (t^2 e^{-t}) dt = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + c$$

$$t = x^2$$

$$t^2 = x^4$$

$$\begin{aligned} \therefore \int 2x^5 e^{-x^2} dx &= -x^4 e^{-x^2} - 2x^2 e^{-x^2} - 2e^{-x^2} + c \\ &= \boxed{e^{-x^2} (-x^4 - 2x^2 - 2) + c} \end{aligned}$$

o.e

$$\text{b) } x \frac{dy}{dx} + 4y = 2x^2 e^{-x^2}$$

$$\div x : \frac{dy}{dx} + \left(\frac{4}{x}\right)y = 2xe^{-x^2}$$

$$I = e^{\int \frac{4}{x} dx} = e^{4 \ln x} = x^4 //$$

$$\underline{x x^4} : x^4 \frac{dy}{dx} + 4x^3 y = 2x^5 e^{-x^2} //$$

$$\therefore \frac{d}{dx} (yx^4) = 2x^5 e^{-x^2}$$

$$y x^4 = \int (2x^5 e^{-x^2}) dx$$

from (a)...  $y x^4 = e^{-x^2} (-x^4 - 2x^2 - 2) + u$

$$\therefore y = \frac{-e^{-x^2}(x^4 + 2x^2 + 2) + u}{x^4}$$

c)  $y=0, x=1$  :  $0 = \frac{-e^{-1}(5) + u}{1}$

$$\therefore u = 5e^{-1} = \frac{5}{e}$$

hence  $y = \frac{-e^{-x^2}(x^4 + 2x^2 + 2) + \frac{5}{e}}{x^4}$