FP3 June 15 kpnine 2

M-A

1. Solve the equation

$$2\cosh^2 x - 3\sinh x = 1$$

giving your answers in terms of natural logarithms.

(6)

$c^2 - s^2 = 1 \implies c^2 = 1 + s^2$

$\therefore 2c^2 - 3s = 1$

$\implies 2(1+s^2) - 3s = 1$

$\therefore 2 + 2s^2 - 3s = 1$

$\therefore 2s^2 - 3s + 1 = 0$

$\therefore (s-1)(2s-1) = 0$

$\implies \sinh x = 1 \implies x = \text{arsinh}\, 1 = \ln\left(1 + \sqrt{2}\right)$

$\sinh x = \frac{1}{2} \implies x = \text{arsinh}\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

2

A curve has equation

$$y = \cosh x, \qquad 1 \leqslant x \leqslant \ln 5$$

Find the exact length of this curve. Give your answer in terms of e.

(5)

$$y = \cosh x \implies \frac{dy}{dx} = \sinh x$$

$$\text{length} = \int_{1}^{\ln 5} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \; \partial x$$

$$c^2 = s^2 = 1$$

$$= \int_{1}^{\ln 5} \sqrt{1 + \sinh^2 x} \; \partial x$$

$$= \int_{1}^{\ln 5} \cosh x \; \partial x$$

$$= \left[\sinh x \right]_{1}^{\ln 5}$$

$$= \sinh(\ln 5) - \sinh(1)$$

$$= \frac{e^{\ln 5} - e^{-\ln 5}}{2} - \frac{e^1 - e^{-1}}{2}$$

$$= \frac{5 - \frac{1}{5}}{2} - \frac{e - \frac{1}{e}}{2}$$

$$= \frac{5 - \frac{1}{5} - e + \frac{1}{e}}{2} \times \frac{5e}{5e} = \frac{25e - e^2 - 5e^2 + 5}{10e}$$

$$= \frac{24e - 5e^2 + 5}{10e}$$

3.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) Find the eigenvalues of A.

(5)

(b) Find a normalised eigenvector for each of the eigenvalues of A.

(5)

(c) Write down a matrix P and a diagonal matrix D such that $P^TAP = D$.

(2)

(a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (2-\lambda)\left((2-\lambda)^2 - 1\right) - (2-\lambda) = 0$

$\Rightarrow (2-\lambda)^3 - (2-\lambda) - (2-\lambda) = 0$

$(2-\lambda)^3 - 2(2-\lambda) = 0$

$(2-\lambda)\left((2-\lambda)^2 - 2\right) = 0$

$(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$

$\Rightarrow 2-\lambda = 0 \therefore \lambda = 2$

$(2-\lambda)^2 - 2 = 0 \Rightarrow 2-\lambda = \pm\sqrt{2}$

$\therefore \lambda = 2 \pm \sqrt{2}$

$\therefore$ E·Values are $\lambda = 2$

$\lambda = 2 \pm \sqrt{2}$

Question 3 continued

(b) Consider $\lambda = 2$:

$Ax = 2x$

$\therefore$
$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$\therefore$
$$\begin{pmatrix} 2x + y \\ x + 2y + z \\ y + 2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \begin{array}{l} \Rightarrow y = 0 \\ \Rightarrow x + z = 0 \\ \Rightarrow y = 0 \end{array}$$

$$x = -z, \quad y = 0$$

Let $x = 1, \quad z = -1, \quad y = 0$

$\therefore$

E.Vector $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Normalised $x = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Consider $\lambda = 2 + \sqrt{2}$:

$$\begin{pmatrix} 2x + z \\ x + 2y + z \\ y + 2z \end{pmatrix} = \begin{pmatrix} 2x + \sqrt{2}\,x \\ 2y + \sqrt{2}\,y \\ 2z + \sqrt{2}\,z \end{pmatrix}$$

$$\Rightarrow \quad y = x\sqrt{2}$$

$$\left.\begin{array}{c} x+z = y\sqrt{2} \\ y = z\sqrt{2} \end{array}\right\}$$

$$y=1 \Rightarrow x=\frac{1}{\sqrt{2}} \Rightarrow z=\frac{1}{\sqrt{2}}$$

$$y=\sqrt{2} \Rightarrow x=1 \Rightarrow z=1$$

$$\therefore \quad x = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

$$\therefore \quad y = x\sqrt{2} = z\sqrt{2}$$

2) $\frac{y}{\sqrt{2}}$ ~~z+z=~~

~~z+z = y\sqrt{2}~~

$x = y\sqrt{2} - \frac{z}{\sqrt{2}}$ $\therefore$ normalise:

$$x = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$x = \frac{y}{\sqrt{2}}$$

$$x = 2 - \sqrt{2}$$

$$\Rightarrow \begin{pmatrix} 2x+y \\ x+2y+z \\ y+2z \end{pmatrix} = \begin{pmatrix} 2x - x\sqrt{2} \\ 2y - y\sqrt{2} \\ 2z - z\sqrt{2} \end{pmatrix}$$

$$\therefore \quad y = -x\sqrt{2}$$

$$x+z = -y\sqrt{2}$$

$$y = -z\sqrt{2}$$

$$\therefore \quad y = -z\sqrt{2} = -x\sqrt{2}$$

$$y = \sqrt{2} \Rightarrow z = -1 \Rightarrow x = -1$$

$$\therefore \quad x = \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \Rightarrow \quad \text{normal } x = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

8

Question 3 continued

$$\therefore \quad \lambda = 2 \Rightarrow x_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda = 2 - \sqrt{2} \Rightarrow x_n = \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$\lambda = 2 + \sqrt{2} \Rightarrow x_n = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

(c)

$$P = \begin{pmatrix} -1/2 & 1/\sqrt{2} & 1/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -1/2 & -1/\sqrt{2} & 1/2 \end{pmatrix}$$

$$P^T A P = \begin{pmatrix} -1/2 & \sqrt{2}/2 & -1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & \sqrt{2}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{-2+\sqrt{2}}{2} \end{pmatrix} \qquad D = \begin{pmatrix} 2-\sqrt{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2+\sqrt{2} \end{pmatrix}$$

(Total 12 marks)

Q3

9

4. The curve C has equation

$$y = \frac{1}{\sqrt{x^2 + 2x - 3}}, \qquad x > 1$$

(a) Find $\int y\,dx$ (3)

The region R is bounded by the curve C, the x-axis and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. (4)

(a). $\int \frac{1}{\sqrt{(x+1)^2 - 4}}\,dx$

$= \text{arcosh}\left(\frac{x+1}{2}\right) + C$

(b) $V = \pi \int_2^3 y^2\,dx$

$V = \pi \int_2^3 \frac{1}{(x+1)^2 - 4}\,dx \qquad \frac{2}{6}\quad \frac{1}{5}$

$= \pi \left[\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|\right]_2^3$

$= \pi \left[\frac{1}{4}\ln\left(\frac{1}{3}\right) - \frac{1}{4}\ln\frac{1}{5}\right]$

$= \frac{1}{4}\pi \ln(5/3)$

5. The points A, B and C have position vectors $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ respectively.

(a) Find a vector equation of the straight line AB.

(2)

(b) Find a cartesian form of the equation of the straight line AB.

(2)

The plane Π contains the points A, B and C.

(c) Find a vector equation of Π in the form $\mathbf{r}.\mathbf{n} = p$.

(4)

(d) Find the perpendicular distance from the origin to Π.

(2)

5(a) $AB = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$

(b) $\therefore r = \lambda \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2\lambda \\ 3-3\lambda \\ 2+\lambda \end{pmatrix}$

$\cancel{x}$ $\dfrac{1-x}{2} = \lambda$

$\dfrac{3-y}{3} = \lambda$

$2-z = \lambda$

$\therefore \quad \dfrac{1-x}{2} = \dfrac{3-y}{3} = 2-z$

$\boxed{x\,6} \Rightarrow 3-3x = 6-2y = \cancel{1}2-6z$

Question 5 continued
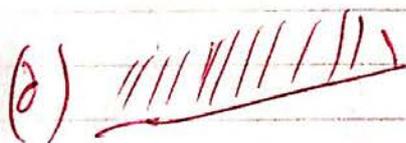
c) $AB = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$

$AC = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\therefore AB \times AC = \begin{matrix} -2 & -3 \\ 1 & -2 \end{matrix} \times \begin{matrix} -1 \\ -2 \end{matrix} \times \begin{matrix} -2 & -3 & -1 \\ 1 & -2 & -2 \end{matrix}$

$= \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$= 8 - 5 = 3$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix} = 3$ $r \cdot \hat{n} = r \hat{n}$

d) //////////////////

$\hat{\underset{\sim}{n}} = \frac{\sqrt{10}}{30} \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$\underset{\sim}{r} \cdot \hat{\underset{\sim}{n}} = \frac{\sqrt{10}}{30} \times 3 = \frac{\sqrt{10}}{10}$

$\therefore d = \frac{\sqrt{10}}{10}$

P 4 4 8 3 3 A 0 1 3 2 8

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6. The hyperbola H is given by the equation $x^2 - y^2 = 1$

(a) Write down the equations of the two asymptotes of H.

(1)

(b) Show that an equation of the tangent to H at the point P (cosh t, sinh t) is

$$y\sinh t = x\cosh t - 1$$

(3)

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR.

(3)

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t.

(3)

6(a). $x = \pm y$ ∴ $\begin{aligned} x &= y \\ x &= -y \end{aligned}$

(b) $\dfrac{\partial y}{\partial x} = \dfrac{\partial y/\partial t}{\partial x/\partial t} = \dfrac{\cosh t}{\sinh t} = \coth(t)$

∴ $y - y_1 = m(x - x_1)$

∴ $y - \sinh t = \dfrac{\cosh t}{\sinh t}x - \dfrac{\cosh^2 t}{\sinh t}$

∴ $y\sinh t - \sinh^2 t = x\cosh t - \cosh^2 t$

∴ $y\sinh t = x\cosh t + \sinh^2 t - \cosh^2 t$

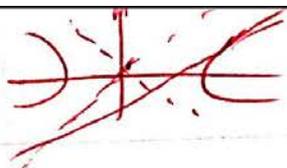∴ $y\sinh t = x\cosh t - (\cosh^2 t - \sinh^2 t)$

∴ $y\sinh t = x\cosh t - 1$ as required.

16

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Question 6 continued

~~(b)~~

(b) Consider $y = x$

$\Rightarrow \quad x \sinh t = x \cosh t - 1$

$\therefore \quad (\sinh t - \cosh t)x = -1$

$\therefore \quad x = \dfrac{1}{\cosh t - \sinh t} = y$

$\therefore$ Let $Q \left(\dfrac{1}{\cosh t - \sinh t} , \dfrac{1}{\cosh t - \sinh t} \right)$

$y = -x$
$\Rightarrow \quad -x \sinh t = x \cosh t - 1$

$\therefore \quad x (\cosh t + \sinh t) = 1$

$\therefore \quad x = \dfrac{1}{\cosh t + \sinh t}$

~~$y =$~~ $\therefore$ Let $R : \left(\dfrac{1}{\cosh t + \sinh t} , \dfrac{-1}{\cosh t + \sinh t} \right)$

Question 6 continued

Midpoint of QR :

$$\left(\frac{1}{2}\left(\frac{1}{\cosh t + \sinh t} + \frac{1}{\cosh t - \sinh t} \right) , \quad \frac{1}{2}\left(\frac{1}{\cosh t - \sinh t} - \frac{1}{c+s} \right) \right)$$

Let $\sinh t = s$ $\qquad$ $c^2 - s^2 = 1$
$\cosh t = c$

Consider x coordinate of Midpoint QR :

$$x = \frac{1}{2}\left(\frac{1}{c+s} + \frac{1}{c-s} \right) = \frac{1}{2}\left(\frac{c-s}{(c+s)(c-s)} + \frac{(c+s)}{(c-s)(c+s)} \right)$$

$$= \frac{1}{2}\left(\frac{2c}{(c+s)(c-s)} \right) = \frac{1}{2}\left(\frac{2c}{c^2 - s^2} \right)$$

$$= \frac{1}{2}\left(\frac{2c}{1} \right) = c = \cosh t$$

$\therefore x$ coordinate $= \cosh t = x_p$

Consider y coordinate of midpoint QR :

$$y = \frac{1}{2}\left(\frac{1}{c-s} - \frac{1}{c+s} \right) = \frac{1}{2}\left(\frac{c+s}{(c-s)(c+s)} - \frac{(c-s)}{(c+s)(c-s)} \right)$$

$$= \frac{1}{2}\left(\frac{c+s-c+s}{c^2-s^2} \right) = \frac{1}{2}\left(\frac{2s}{c^2-s^2} \right) = \frac{1}{2}\left(\frac{2s}{1} \right)$$
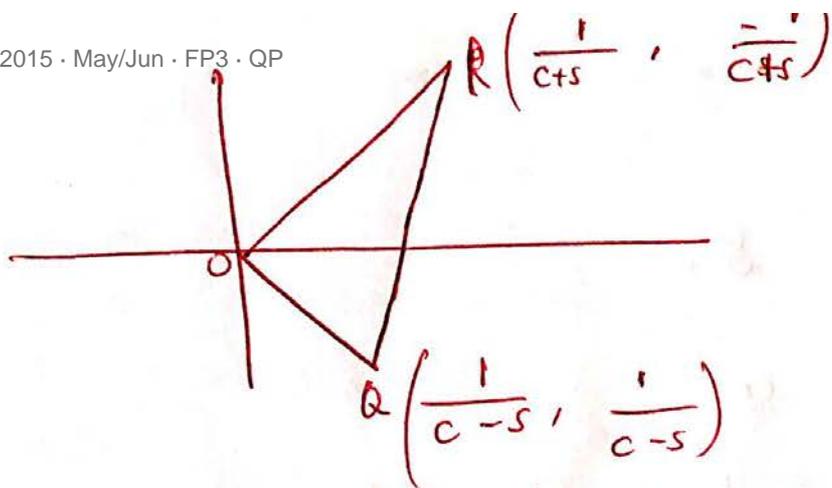
P 4 4 8 3 3 A 0 1 8 2 8

Question 6 continued

$$= s \neq \sinh t = y_p$$

$$\therefore \quad x_p = \cot$$

$$\therefore \quad P \text{ is midpoint of } QR$$

(d)

$R\left(\dfrac{1}{c+s}, \dfrac{-1}{c+s}\right)$

$Q\left(\dfrac{1}{c-s}, \dfrac{1}{c-s}\right)$

Let $c = \cosh t$
Let $s = \sinh t$

Area $= \frac{1}{2} \times |OQ| \times |OR|$

$= \dfrac{1}{2} \cdot \sqrt{\left(\dfrac{1}{c-s}\right)^2 + \left(\dfrac{1}{(c-s)^2}\right)} \quad \times \quad \sqrt{\dfrac{1}{(c+s)^2} + \dfrac{1}{(c+s)^2}}$

$= \dfrac{1}{2} \cdot \sqrt{2} \cdot \sqrt{\dfrac{1}{(c-s)^2}} \cdot \sqrt{2} \sqrt{\dfrac{1}{(c+s)^2}}$

$= \dfrac{1}{2} \cdot \sqrt{2} \cdot \dfrac{1}{c-s} \cdot \sqrt{2} \cdot \dfrac{1}{c+s}$

$= \dfrac{1}{(c-s)(c+s)} = \dfrac{1}{c^2 - s^2} = 1$

∴ Area is always one regardless of how t varies

∴ Area is independent of t

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7.

$$I_n = \int \sin^n x \, dx, \quad n \geqslant 0$$

(a) Prove that for $n \geqslant 2$

$$I_n = \frac{1}{n}(-\sin^{n-1}x \cos x + (n-1)I_{n-2})$$

(4)

Given that n is an odd number, $n \geqslant 3$

(b) show that

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{(n-1)(n-3)...6.4.2}{n(n-2)(n-4)...7.5.3}$$

(4)

(c) Hence find $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \, dx$

(3)

7(a) $\quad I_n = \int \sin^n x \, dx = \int \sin^{n-1} \sin x \, dx$

Let $u = \sin^{n-1} x \qquad u' = (n-1)(\sin^{n-2} x) \cos x$

$\quad v' = \sin x \qquad\qquad v = -\cos x$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \, dx$

$\qquad\qquad\qquad$ since $\cos^2 x = 1 - \sin^2 x$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x - \sin^n x \, dx$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)\left[I_{n-2} - I_n \right]$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n$

$\therefore (1+n-1)I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$

$\therefore n I_n = -\cos x \sin^{n-1} x + (n-1)I_{n-2}$

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$$\therefore \quad I_n = \frac{1}{n}\left(-\cos x \, \sin^{n-1}x + (n-1)\, I_{n-2}\right)$$

(b) Let $I_n = \displaystyle\int_0^{\pi/2} \sin^n x \, dx$

$$I_n = \frac{1}{n}\left(\Big[-\cos x \, \sin^{n-1}x\Big]_0^{\pi/2} + (n-1)\, I_{n-2}\right)$$

$$\therefore \quad I_n = \frac{(n-1)}{n}\, I_{n-2} \quad, \quad I_{n-2} = \frac{n-3}{n-2}\, I_{n-4}, \quad \cdots$$

$$\therefore \quad I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) I_{n-4} = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) I_{n-6} \cdots$$

$$\Rightarrow \quad I_n = \frac{(n-1)(n-3)(n-5)\, \cdots \, 6 \cdot 4 \cdot 2}{n(n-2)(n-4)\, \cdots \, 7 \cdot 5 \cdot 3}$$

as required

Question 7 continued

(C) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos^2 x \, dx$

$$= \int_0^{\pi/2} \sin^5 x \left(1 - \sin^2 x\right) dx$$

$$= \int_0^{\pi/2} \sin^5 x - \sin^7 x \, dx$$

$$= \int_0^{\pi/2} \sin^5 x \, dx - \int_0^{\pi/2} \sin^7 x \, dx$$

$$= \frac{4 \cdot 2 \cdot 1}{5 \cdot 3 \cdot 1} - \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1}$$

$$= \frac{8}{15} - \frac{16}{35} = \frac{8}{105}$$

8. The ellipse E has equation $x^2 + 4y^2 = 4$

    (a) (i) Find the coordinates of the foci, F_1 and F_2, of E.

       (ii) Write down the equations of the directrices of E.

           (4)

    (b) Given that the point P lies on the ellipse, show that

$$|PF_1| + |PF_2| = 4$$

           (4)

A chord of an ellipse is a line segment joining two points on the ellipse.

The set of midpoints of the parallel chords of E with gradient m, where m is a constant, lie on a straight line l.

    (c) Find an equation of l.

           (6)

8(a). $\dfrac{x^2}{4} + \dfrac{y^2}{} = 1$

$\cancel{b^2} = 1 = 4(1 - e^2)$

$\therefore \dfrac{1}{4} = 1 - e^2 \implies e^2 = 3/4$

$\therefore e = \dfrac{\sqrt{3}}{2}$

(i) $\cancel{f}$ $F_1 \left(\sqrt{3}, 0 \right)$

    $F_2 \left(-\sqrt{3}, 0 \right)$

(ii) $x = \pm \dfrac{2}{\frac{\sqrt{3}}{2}} = \pm \dfrac{4\sqrt{3}}{3}$

Question 8 continued

b)
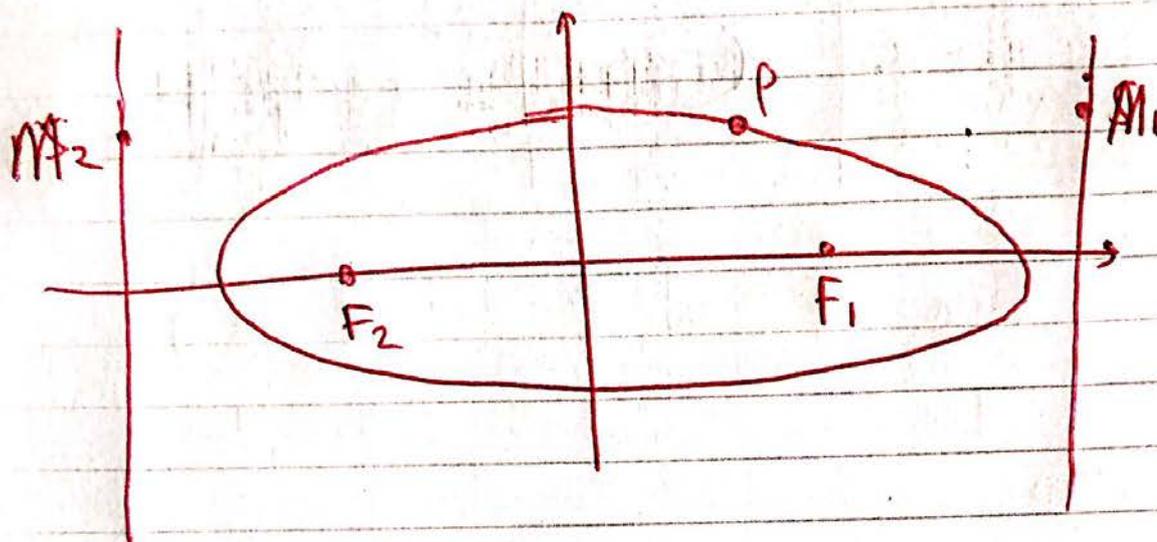
$$e = \frac{PF_1}{PM_1} \quad , \quad e = \frac{PF_2}{PM_2}$$

$$\therefore \quad PF_1 = ePM_1 \quad \& \quad PF_2 = ePM_2$$

$$PM_1 + PM_2 = \frac{4\sqrt{3}}{3} + \frac{4\sqrt{3}}{3} = \frac{8\sqrt{3}}{3}$$

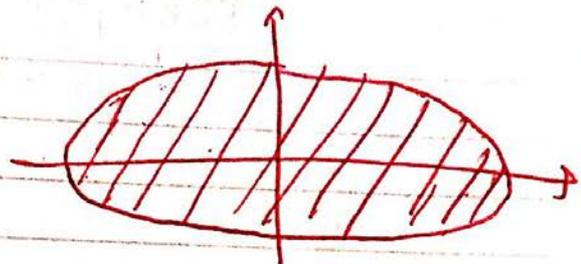$$\therefore \quad ePM_1 + ePM_2 = e \times \frac{8\sqrt{3}}{3}$$

$$\therefore \quad PF_1 + PF_2 = e \times \frac{8\sqrt{3}}{3} = \frac{\sqrt{3}}{2} \times \frac{8\sqrt{3}}{3}$$

$$\therefore \quad PF_1 + PF_2 = 4$$

as required

(C) $x^2 + 4y^2 = 4$

$\therefore \quad 2x + 8y \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{-2x}{8y} = -\frac{x}{4y}$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{4y} = m$

$\therefore \quad -x = 4ym$

$\therefore \quad y = -\frac{x}{4m}$