

## FP3 UK June 2016 Model Answers

Kprime2

Leave blank

1.

$$A = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix A is singular, find the possible values of k.

(4)

1.

$$\det A = -2 \begin{vmatrix} 1 & 3 \\ -1 & k \end{vmatrix} - \begin{vmatrix} k & 3 \\ 2 & k \end{vmatrix} - 3 \begin{vmatrix} k & 1 \\ 2 & -1 \end{vmatrix}$$

$$= -2(k+3) - (k^2-6) - 3(-k-2)$$

$$= -2k - 6 - k^2 + 6 + 3k + 6$$

$$= -k^2 + k + 6 = 0$$

$$\therefore k^2 - k - 6 = 0$$

$$\therefore (k-3)(k+2) = 0$$

$$\therefore k = \underline{\underline{3}}$$

$$k = \underline{\underline{-2}}$$



Leave  
blank2. The curve  $C$  has equation

$$y = \frac{x^2}{8} - \ln x, \quad 2 \leq x \leq 3$$

Find the length of the curve  $C$  giving your answer in the form  $p + \ln q$ , where  $p$  and  $q$  are rational numbers to be found.

(7)

$$2. \quad y = \frac{x^2}{8} - \ln x$$

$$\frac{\partial y}{\partial x} = \frac{x}{4} - \frac{1}{x} = \frac{x^2}{4x} - \frac{4}{4x}$$

$$= \frac{x^2 - 4}{4x}$$

$$\therefore \left( \frac{\partial y}{\partial x} \right)^2 = \frac{(x^2 - 4)^2}{16x^2}$$

$$\therefore 1 + \left( \frac{\partial y}{\partial x} \right)^2 = 1 + \frac{(x^2 - 4)^2}{16x^2} = \frac{16x^2 + x^4 - 8x^2 + 16}{16x^2}$$

$$= \frac{x^4 + 8x^2 + 16}{16x^2} = \frac{(x^2 + 4)^2}{16x^2}$$

$$\therefore \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} = \frac{\cancel{16x^2} (x^2 + 4)}{\cancel{16x^2}} = \frac{x^2 + 4}{4x}$$

$$\therefore S = \int_2^3 \sqrt{1 + \left(\frac{dx}{dx}\right)^2} dx$$

$$= \int_2^3 \frac{x^2 + 4}{4x} dx$$

$$= \int_2^3 \left( \frac{1}{4}x + \frac{1}{x} \right) dx$$

$$= \left[ \frac{1}{8}x^2 + \ln x \right]_2^3$$

$$= \frac{9}{8} + \ln 3 - \frac{1}{2} - \ln 2$$

$$= \frac{5}{8} + \ln \frac{3}{2}$$

Leave  
blank

3. (a) Prove that

$$\frac{d(\operatorname{arcoth} x)}{dx} = \frac{1}{1-x^2} \quad (3)$$

Given that  $y = (\operatorname{arcoth} x)^2$ ,

(b) show that

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} = \frac{k}{1-x^2}$$

where  $k$  is a constant to be determined.

(5)

3(a). Let  $y = \operatorname{arcoth} x$ 

$$\frac{\cancel{\partial y}}{\partial x} = \therefore \coth y = x$$

$$\therefore \frac{\partial y}{\partial x} (-\operatorname{cosech}^2 y) = 1$$

$$\operatorname{cosech}^2 y \equiv \coth^2 y - 1$$

$$\therefore \frac{\partial y}{\partial x} \cdot (-(\coth^2 y - 1)) = 1$$

$$\coth y = x \Rightarrow \coth^2 y = x^2$$

$$\therefore \frac{\partial y}{\partial x} \cdot (-(x^2 - 1)) = 1$$

$$\therefore \frac{\partial y}{\partial x} = \frac{1}{-(x^2 - 1)} = \frac{1}{1 - x^2}$$

as  
required.

Question 3 continued

$$(b) \quad y = (\operatorname{arccoth} x)^2$$

$$\therefore \frac{dy}{dx} = \frac{2 \operatorname{arccoth} x}{1-x^2} \Rightarrow -2x \frac{dy}{dx} = \frac{-4x \operatorname{arccoth} x}{1-x^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(1-x^2) \left( \frac{2}{1-x^2} \right) - 2 \operatorname{arccoth} x (-2x)}{(1-x^2)^2}$$

$$= \frac{2 + 4x \operatorname{arccoth} x}{(1-x^2)^2}$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} = (1-x^2) \left( \frac{2 + 4x \operatorname{arccoth} x}{(1-x^2)^2} \right)$$

$$= \frac{2 + 4x \operatorname{arccoth} x}{1-x^2}$$

$$\therefore \text{LHS} = (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} = \frac{2 + 4x \operatorname{arccoth} x}{1-x^2} - \frac{4x \operatorname{arccoth} x}{1-x^2}$$

$$= \frac{2 + 4x \operatorname{arccoth} x - 4x \operatorname{arccoth} x}{1-x^2} = \frac{2}{1-x^2}$$

$$\underline{\underline{k=2}}$$



4. (i) Find, without using a calculator,

$$\int_3^5 \frac{1}{\sqrt{15+2x-x^2}} dx$$

giving your answer as a multiple of  $\pi$ .

(5)

(ii)

(a) Show that

$$5 \cosh x - 4 \sinh x = \frac{e^{2x} + 9}{2e^x}$$

(3)

(b) Hence, using the substitution  $u = e^x$  or otherwise, find

$$\int \frac{1}{5 \cosh x - 4 \sinh x} dx$$

(4)

$$\begin{aligned} 4(i). \quad 15+2x-x^2 &= -(x^2-2x-15) \\ &= -(x-1)^2-16 \\ &= 16-(x-1)^2 \end{aligned}$$

$$\therefore \int_3^5 \frac{1}{\sqrt{16-(x-1)^2}} dx = \left[ \arcsin \left( \frac{x-1}{4} \right) \right]_3^5$$

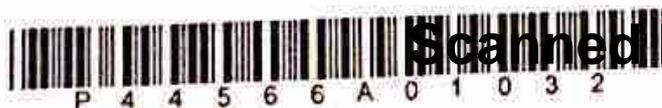
$$= \arcsin(1) - \arcsin \frac{1}{2}$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



Leave  
blank

Question 4 continued

$$(ii) \text{ LHS} = 5 \cosh x - 4 \sinh x = \frac{5e^x + 5e^{-x}}{2} - \frac{4e^x - 4e^{-x}}{2}$$

$$= \frac{5e^x + 5e^{-x} - 4e^x + 4e^{-x}}{2}$$

$$= \frac{e^x + 9e^{-x}}{2} = \frac{(e^x + 9e^{-x})e^x}{2e^x}$$

$$= \frac{e^{2x} + 9}{2e^x} = \underline{\underline{\text{RHS}}}$$
 as required.

$$(b) u = e^x \Rightarrow \frac{du}{dx} = e^x = u \Rightarrow dx = \frac{1}{u} \cdot du$$

$$\therefore \int \frac{1}{5 \cosh x - 4 \sinh x} dx = \int \frac{2e^x}{e^{2x} + 9} dx$$

$$= \int \frac{2u}{u^2 + 9} \cdot \frac{1}{u} du$$

$$= \int \frac{2}{u^2 + 9} du = \frac{2}{3} \arctan\left(\frac{u}{3}\right) + C$$

$$= \underline{\underline{\frac{2}{3} \arctan\left(\frac{e^x}{3}\right) + C}}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



Leave  
blank5. The hyperbola  $H$  has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

The point  $P(4 \sec \theta, 3 \tan \theta)$ ,  $0 < \theta < \frac{\pi}{2}$ , lies on  $H$ .(a) Show that an equation of the normal to  $H$  at the point  $P$  is

$$3y + 4x \sin \theta = 25 \tan \theta \quad (5)$$

The line  $l$  is the directrix of  $H$  for which  $x > 0$ The normal to  $H$  at  $P$  crosses the line  $l$  at the point  $Q$ . Given that  $\theta = \frac{\pi}{4}$ (b) find the  $y$  coordinate of  $Q$ , giving your answer in the form  $a + b\sqrt{2}$ , where  $a$  and  $b$  are rational numbers to be found.

(6)

$$5(a). \text{ @ } P, \frac{dy}{dx} = \frac{\frac{dx}{d\theta}}{\frac{dx}{d\theta}} = \frac{3 \sec^2 \theta}{4 \sec \theta \tan \theta}$$

$$= \frac{3 \sec \theta}{4 \tan \theta} = \frac{3}{4 \sin \theta}$$

$$\therefore \text{gradient of normal} = -\frac{4 \sin \theta}{3}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 3 \tan \theta = -\frac{4 \sin \theta}{3}(x - 4 \sec \theta)$$

$$\frac{y - 3 \tan \theta}{4 \sin \theta} = \frac{3}{4 \sin \theta} (x - 4 \sec \theta)$$

$$\therefore y - 3 \tan \theta = -\frac{4 \sin \theta}{3} (x - 4 \sec \theta)$$

$$\therefore 3y - 9 \tan \theta = -4 \sin \theta x + 16 \tan \theta$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



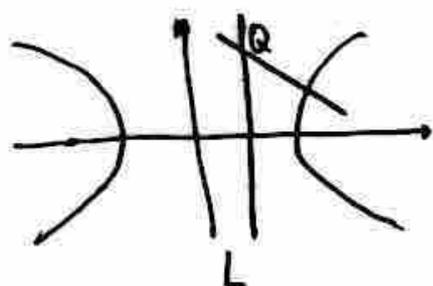
## Question 5 continued

$$\therefore 3y + 4x \sin \theta = 25 \tan \theta$$

as required.

$$(b) \begin{cases} a^2 = 16 \\ b^2 = 9 \end{cases} \quad \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} 9 = 16(e^2 - 1) \\ \therefore e = \frac{5}{4} \end{array}$$

$$\theta = \frac{\pi}{4} \Rightarrow 3y + 2\sqrt{2}x = 25$$



$$\text{directrix } \frac{a}{e} = \frac{16}{5}$$

$$x_Q = \frac{16}{5}$$

$$\Rightarrow 3y + \frac{32}{5}\sqrt{2} = 25$$

$$\therefore y_Q = \frac{25}{3} - \frac{32}{15}\sqrt{2}$$



Leave  
blank

6.

$$M = \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix}$$

where  $p$  and  $q$  are constants.

Given that  $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$  is an eigenvector of the matrix  $M$ ,

(a) find the eigenvalue corresponding to this eigenvector, (3)

(b) find the value of  $p$  and the value of  $q$ . (3)

Given that 6 is another eigenvalue of  $M$ ,

(c) find a corresponding eigenvector. (2)

Given that  $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$  is a third eigenvector of  $M$  with eigenvalue 3

(d) find a matrix  $P$  and a diagonal matrix  $D$  such that

$$P^{-1}MP = D \quad (3)$$

$$6(a). \quad \begin{pmatrix} p & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & q \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2p+4 \\ -18 \\ 4+q \end{pmatrix} = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix} \Rightarrow \begin{matrix} -18 = -2\lambda \\ \therefore \lambda = 9 \end{matrix}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



## Question 6 continued

$$(b) \quad 2p + q = 2\lambda \Rightarrow 2p + q = 18 \\ \therefore p = 7$$

$$4 + q = 9 \Rightarrow q = 5$$

$$(c) \quad \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7x - 2y \\ -2x + 6y - 2z \\ -2y + 5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

$$\left. \begin{array}{l} 7x - 2y = 6x \Rightarrow x = 2y \\ -2x + 6y - 2z = 6y \Rightarrow z = -x \\ -2y + 5z = 6z \Rightarrow z = -2y \end{array} \right\} \Rightarrow \underline{x = 2y = -z}$$

$$\text{Let } x = 2 \Rightarrow y = 1 \Rightarrow z = -2$$

$$\therefore \text{E-vector is } \underline{\underline{\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}}$$

## Question 6 continued

$$(d) \lambda = 9 \text{ \& } \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{Normalise} \Rightarrow \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda = 6 \text{ \& } \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\text{Normalise} \Rightarrow \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\lambda = 3 \text{ \& } \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ Normalise} \Rightarrow \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\therefore P = \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -2 & -2 & 2 \\ 1 & -2 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



Leave  
blank

7. Given that

$$I_n = \int \frac{\sin nx}{\sin x} dx, \quad n \geq 1$$

(a) prove that, for  $n \geq 3$ 

$$I_n - I_{n-2} = \int 2 \cos(n-1)x dx \quad (3)$$

(b) Hence, showing each step of your working, find the exact value of

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{\sin 5x}{\sin x} dx$$

giving your answer in the form  $\frac{1}{12}(a\pi + b\sqrt{3} + c)$ , where  $a$ ,  $b$  and  $c$  are integers to be found. (7)

$$\begin{aligned} I_n - I_{n-2} &= \int \frac{\sin nx}{\sin x} dx - \int \frac{\sin[(n-2)x]}{\sin x} dx \\ &= \int \frac{\sin(nx) - \sin(nx-2x)}{\sin x} dx \\ &= \int \frac{2 \cos\left(\frac{2nx-2x}{2}\right) \sin\left(\frac{2x}{2}\right)}{\sin x} dx \quad * \\ &= \int 2 \cos(n-1)x dx \end{aligned}$$

\* Using  $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



## Question 7 continued

$$\begin{aligned}
 I_5 &= I_3 = 2 \int_{\pi/12}^{\pi/6} \cos 4x \, dx \\
 &= 2 \left[ \frac{1}{4} \sin 4x \right]_{\pi/12}^{\pi/6} \\
 &= 0 \Rightarrow I_5 = I_3
 \end{aligned}$$

$$\begin{aligned}
 I_3 &= I_1 = \int_{\pi/12}^{\pi/6} 2 \cos 2x \, dx \\
 &= \left[ \sin 2x \right]_{\pi/12}^{\pi/6} \\
 &= \frac{-1 + \sqrt{3}}{2}
 \end{aligned}$$

$$I_1 = \int_{\pi/12}^{\pi/6} 1 \, dx = \frac{\pi}{6} - \frac{\pi}{12} = \frac{\pi}{12}$$

$$\therefore I_3 = \frac{\pi}{12} + \frac{-1 + \sqrt{3}}{2} = \frac{\pi}{12} + \frac{-6 + 6\sqrt{3}}{12}$$

$$\Rightarrow I_5 = \frac{1}{12} (\pi + 6\sqrt{3} - 6) \quad \begin{array}{l} a=1 \\ b=6 \\ c=-6 \end{array}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



Leave  
blank8. The plane  $\Pi_1$  has equation

$$x - 5y - 2z = 3$$

The plane  $\Pi_2$  has equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

where  $\lambda$  and  $\mu$  are scalar parameters.

- (a) Show that  $\Pi_1$  is perpendicular to  $\Pi_2$ . (4)
- (b) Find a cartesian equation for  $\Pi_2$ . (2)
- (c) Find an equation for the line of intersection of  $\Pi_1$  and  $\Pi_2$  giving your answer in the form  $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are constant vectors to be found. (6)

$$8(a) \quad \Pi_2: \quad \mathbf{n}_2 = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 3 \\ 2 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix}$$

$$\Pi_1: \quad \mathbf{n}_1 = \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix}$$

If  $\Pi_1$  &  $\Pi_2$  are perpendicular then their normals must be perpendicular.

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ -2 \end{pmatrix} = 7 - 25 + 18 = 0$$

$\Rightarrow \Pi_1$  &  $\Pi_2$  are perpendicular.



Leave  
blank

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

## Question 8 continued

$$(b) \quad \underline{L} \cdot \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 5 \\ -9 \end{pmatrix}$$

$$= 8$$

$$\Rightarrow \underline{\underline{7x + 5y - 9z = 8}}$$

$$(c) \quad x - 5y - 2z = 3 \Rightarrow 5y = x - 2z - 3$$

$$7x + 5y - 9z = 8$$

$$\therefore 7x + x - 2z - 3 - 9z = 8$$

$$\therefore 8x - 11z = 11$$

$$\therefore \underline{\underline{x = \frac{11}{8} + \frac{11}{8}z}}$$

$$\therefore 5y = \frac{11}{8} + \frac{11}{8}z - 2z - 3$$

$$5y = -\frac{13}{8} - \frac{5}{8}z$$

$$\therefore \underline{\underline{y = -\frac{13}{40} - \frac{1}{8}z}}$$

$$z = z$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{11}{8} \\ -\frac{13}{40} \\ 0 \end{pmatrix} + z \begin{pmatrix} \frac{11}{8} \\ -\frac{1}{8} \\ 1 \end{pmatrix}$$



Leave  
blank

## Question 8 continued

$$= \begin{pmatrix} 11/8 \\ -13/40 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ -1 \\ 8 \end{pmatrix}$$

$$\therefore \left( \begin{matrix} \sim \\ - \end{matrix} \begin{pmatrix} 11/8 \\ -13/40 \\ 0 \end{pmatrix} \right) \times \begin{pmatrix} 11 \\ -1 \\ 8 \end{pmatrix} = 0$$

Note: There are many possible solutions for this part of the question.

Solutions will vary depending on what you expressed  $v_{x,y,z}^{\text{your}}$  in terms of ...