

FP3 June 2017 (MA)

Q1) let $y = \operatorname{arsinh}(t)$

$$t = \tanh x$$
$$\frac{dt}{dx} = \operatorname{sech}^2 x$$

then $\frac{dy}{dt} = \frac{1}{\sqrt{1+t^2}}$

 \therefore

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1+t^2}} = \frac{\operatorname{sech}^2 x}{\sqrt{1+\tanh^2 x}}$$

Q2a) $x = 6 \cos \theta$
 $\frac{dx}{d\theta} = -6 \sin \theta$

$y = 5 \sin \theta$
 $\frac{dy}{d\theta} = 5 \cos \theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5 \cos \theta}{-6 \sin \theta} \therefore m_c = \frac{6 \sin \theta}{5 \cos \theta} //$$

(perpendicular).

$$y - 5 \sin \theta = \frac{6 \sin \theta}{5 \cos \theta} (x - 6 \cos \theta)$$

$$y - 5 \sin \theta = \left(\frac{6 \sin \theta}{5 \cos \theta} \right) x - \frac{36 \sin \theta}{5}$$

$$\underline{\times 5 \cos \theta} : 5y \cos \theta - 25 \sin \theta \cos \theta = 6x \sin \theta - 36 \sin \theta \cos \theta$$

$$\Rightarrow 6x \sin \theta - 5y \cos \theta = (36 - 25) \sin \theta \cos \theta$$

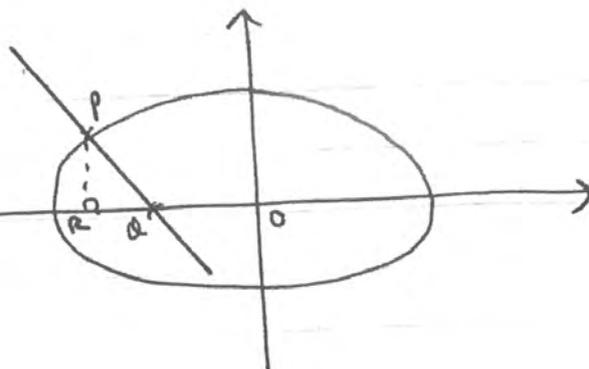
$$\Rightarrow 6x \sin \theta - 5y \cos \theta = 11 \sin \theta \cos \theta$$

$$b) \underline{y=0} : 6x \sin \theta = 11 \sin \theta \cos \theta$$

$$(x > 0) \quad x = \frac{11}{6} \cos \theta = OQ$$

$$\text{so } OQ = \frac{11}{6} \cos \theta$$

$$\text{and } OR = 6 \cos \theta$$



$$\frac{OQ}{OR} = \frac{\frac{11}{6} \cos \theta}{6 \cos \theta} = \frac{11}{36}$$

$$\underline{\text{finding } e^2} : b^2 = a^2 (1 - e^2)$$

$$25 = 36 (1 - e^2)$$

$$\frac{25}{36} = 1 - e^2$$

$$\therefore e^2 = 1 - \frac{25}{36} = \frac{11}{36} = \frac{OQ}{OR}$$



$$Q3a) \quad LHS = \cosh 2x = \frac{e^{2x} + e^{-2x}}{2}$$

$$RHS = 2 \cosh^2 x - 1 = 2 \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) - 1$$

$$= \left(\frac{e^{2x} + 2 + e^{-2x}}{2} \right) - 1$$

$$= \frac{e^{2x} + 2 + e^{-2x} - 2}{2} = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh 2x = LHS$$

$$b) \quad 29 \cosh x - 3(2 \cosh^2 x - 1) = 38$$

$$29 \cosh x - 6 \cosh^2 x + 3 = 38$$

$$6 \cosh^2 x - 29 \cosh x + 35 = 0$$

By Quadratic Formula, $\cosh x = \frac{5}{2}$, $\cosh x = \frac{7}{3}$

$$\Rightarrow \frac{e^x + e^{-x}}{2} = \frac{5}{2}$$

$$\times 2e^x \Rightarrow e^{2x} + 1 = 5e^x$$

$$\Rightarrow e^{2x} - 5e^x + 1 = 0$$

$$e^x = \frac{5 \pm \sqrt{21}}{2}$$

$$x = \ln \left(\frac{5 + \sqrt{21}}{2} \right), \quad x = \ln \left(\frac{5 - \sqrt{21}}{2} \right)$$

$$\Rightarrow \frac{e^x + e^{-x}}{2} = \frac{7}{3}$$

$$\Rightarrow e^{2x} + 1 = \frac{14}{3} e^x$$

$$\Rightarrow e^{2x} - \frac{14}{3} e^x + 1 = 0$$

$$e^x = \frac{7 \pm 2\sqrt{10}}{3}$$

$$\boxed{x = \ln\left(\frac{7+2\sqrt{10}}{3}\right)}, \quad \boxed{x = \ln\left(\frac{7-2\sqrt{10}}{3}\right)}$$

Q4)

$$\int_{-1}^7 \left[\frac{(x+2)^{\frac{1}{2}}}{x+5} \right] dx$$

$$\Downarrow$$

$$\int_1^3 \left[\frac{u}{u^2+3} \times 2u \right] du$$

$$\Rightarrow 2 \int_1^3 \left[\frac{u^2}{u^2+3} \right] du$$

$$\Rightarrow 2 \int_1^3 \left[\frac{u^2+3-3}{u^2+3} \right] du = 2 \int_1^3 \left[1 - \frac{3}{u^2+3} \right] du$$

$$\Rightarrow 2 \left[u - \frac{3}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) \right]_1^3$$

↑
From formula book.

$$\Rightarrow 2 \left[3 - \frac{3}{\sqrt{3}} \arctan\left(\frac{3}{\sqrt{3}}\right) \right] - 2 \left[1 - \frac{3}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$\Rightarrow 2 \left[3 - \frac{\pi}{\sqrt{3}} \right] - 2 \left[1 - \frac{\pi}{2\sqrt{3}} \right]$$

$$= 6 - 2 + 2 \left(\frac{\pi}{2\sqrt{3}} - \frac{\pi}{\sqrt{3}} \right)$$

$$= 4 + 2 \left(\frac{-\pi}{2\sqrt{3}} \right)$$

$$= 4 - \frac{\pi}{\sqrt{3}} = \boxed{4 - \frac{\sqrt{3}\pi}{3}}$$

$$\text{Q5 a) } \pi_1: r \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 5$$

$$\pi_2: r \cdot \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = 7$$

using normals: $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = 6 - 2 + 12 = 16$

$$\left| \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\left| \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} \right| = \sqrt{6^2 + 1^2 + 4^2} = \sqrt{53}$$

$$\therefore \cos \theta = \frac{16}{\sqrt{14 \times 53}}$$

$$\cos \theta = \frac{16}{\sqrt{742}}$$

$$\text{so } \theta = \cos^{-1} \left(\frac{16}{\sqrt{742}} \right) = \boxed{54^\circ}$$

b) l is perp. to Π_1 \therefore direction vector: $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

and passes through $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$\therefore r_l = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} //$$

sub l into Π_2 :

$$\begin{pmatrix} 2 + \lambda \\ 3 - 2\lambda \\ -1 - 3\lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = 7$$

$$\Rightarrow 12 + 6\lambda + 3 - 2\lambda + 4 + 12\lambda = 7$$

$$\Rightarrow 16\lambda = -12$$

$$\therefore \lambda = -\frac{3}{4} \rightarrow Q = \begin{pmatrix} 2 - 3/4 \\ 3 - 2(-3/4) \\ -1 - 3(-3/4) \end{pmatrix}$$

c) to find the normal of Π_3 , we need to 'cross' the normals of Π_1 and Π_2 .

This will give us a vector perpendicular to Π_3 which we need.

$$Q = \begin{pmatrix} 5/4 \\ 9/2 \\ 5/4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 6 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 11 \\ -14 \\ 13 \end{pmatrix}$$

(By calculator).

$$\text{so } r \cdot \begin{pmatrix} 11 \\ -14 \\ 13 \end{pmatrix} = \begin{pmatrix} 11 \\ -14 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} 5/4 \\ 9/2 \\ 5/4 \end{pmatrix} = 11(5/4) - 14(9/2) + 13(5/4)$$

$$\Rightarrow r \cdot \begin{pmatrix} 11 \\ -14 \\ 13 \end{pmatrix} = -33 //$$

$$\text{Q6a)} \quad M = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}$$

$$\det M = 1 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - k \begin{vmatrix} 2 & 1 \\ -4 & -1 \end{vmatrix} + 0 \begin{vmatrix} 2 & -2 \\ -4 & 1 \end{vmatrix}$$

$$= -2(-1) - 1 - k(-2 + 4)$$

$$= 2 - 1 + 2k - 4k = \frac{1 - 2k}{\square}$$

$$\text{b)} \quad M^T = \begin{pmatrix} 1 & 2 & -4 \\ k & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (\text{Transpose})$$

$$\text{Matrix of minors} = \begin{pmatrix} \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} k & -1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} k & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -4 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ k & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ k & -2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -k & k \\ 2 & -1 & 1 \\ -6 & 1+4k & -2-2k \end{pmatrix} //$$

changing signs
according to rule
of alternating signs:

$$\begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -1-4k & -2-2k \end{pmatrix} //$$

$$\& \det M = 1 - 2k$$

$$\text{hence } M^{-1} = \frac{1}{1-2k} \begin{pmatrix} 1 & k & k \\ -2 & -1 & -1 \\ -6 & -1-4k & -2-2k \end{pmatrix}$$

c) $TL_1 = L_2$. $L_2: r = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} //$

$$T^{-1}TL_1 = T^{-1}L_2$$

$$L_1 = T^{-1}L_2$$

T is just M with $k=0$.

$$\therefore T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix}$$

$$\therefore L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & -1 \\ -6 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1+5\lambda \\ 2\lambda-2 \\ 3+\lambda \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 + 5\lambda + 0 \\ -2 - 10\lambda + 2 - 2\lambda - 3 - \lambda \\ -6 - 30\lambda + 2 - 2\lambda - 6 - 2\lambda \end{pmatrix} = \begin{pmatrix} 1 + 5\lambda \\ -3 - 13\lambda \\ -10 - 34\lambda \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 \\ -3 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -13 \\ -34 \end{pmatrix}$$

Cartesian eqn: $\frac{x-1}{5} = \frac{y+3}{-13} = \frac{z+10}{-34}$

$$(Q7a) \quad I_n = \int_0^{\ln 2} [\cosh^n x] dx = \int_0^{\ln 2} [\cosh^{n-2} x \sinh x]$$

$$= \int_0^{\ln 2} [\cosh^{n-2} x] [\cosh^2 x] dx$$

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \therefore \cosh^2 x &= 1 + \sinh^2 x \end{aligned}$$

$$\Rightarrow \int_0^{\ln 2} [\cosh^{n-2} x] [1 + \sinh^2 x] dx$$

$$\Rightarrow \int_0^{\ln 2} [\cosh^{n-2} x] dx + \int_0^{\ln 2} [\cosh^{n-2} x \sinh^2 x] dx$$

$$\therefore I_n = I_{n-2} + \int_0^{\ln 2} [\cosh^{n-2} x \sinh x] [\sinh x] dx$$

By Parts

$$\left[\frac{dv}{dx} = \cosh^{n-2} x \sinh x \rightarrow v = \frac{\cosh^{n-1} x}{n-1} \right] \left[u = \sinh x \right]$$

(Pattern!) $\left[\therefore u' = \cosh x \right]$

$$\therefore I_n = I_{n-2} + \left[\frac{\sinh x \cosh^{n-1} x}{n-1} \right]_0^{\ln 2} - \frac{1}{n-1} \int_0^{\ln 2} (\cosh^{n-1} x \cosh x) dx$$

$$I_n = I_{n-2} + \left[\frac{\frac{3}{4} \left(\frac{5}{4}\right)^{n-1}}{n-1} \right] - \frac{1}{n-1} \int_0^{\ln 2} [\cosh^n x] dx$$

$$\therefore I_n = I_{n-2} + \frac{3 \left(\frac{5}{4}\right)^{n-1}}{4(n-1)} - \frac{1}{n-1} I_n$$

$$I_n \left(1 + \frac{1}{n-1}\right) = I_{n-2} + \frac{3 \left(\frac{5}{4}\right)^{n-1}}{4(n-1)}$$

$$\left(\frac{n-1+1}{n-1}\right) I_n = \left(\frac{n}{n-1}\right) I_n = I_{n-2} + \frac{3 \left(\frac{5}{4}\right)^{n-1}}{4(n-1)}$$

$$\times \frac{(n-1)}{n}: I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3 \left(\frac{5}{4}\right)^{n-1}}{4n}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3(5)^{n-1}}{(4)^{n-1+1} n}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} + \frac{3(5)^{n-1}}{(4)^n n}$$

$$b) \int_0^{\ln 2} \cosh^4 x \, dx = I_4$$

$$I_4 = \frac{(4-1)}{4} I_2 + \frac{3(s)^{4-1}}{4(4)^4}$$

$$I_4 = \frac{3}{4} I_2 + \frac{375}{1024}$$

$$I_2 = \frac{1}{2} I_0 + \frac{3(s)}{4^2(2)} = \frac{1}{2} I_0 + \frac{15}{32}$$

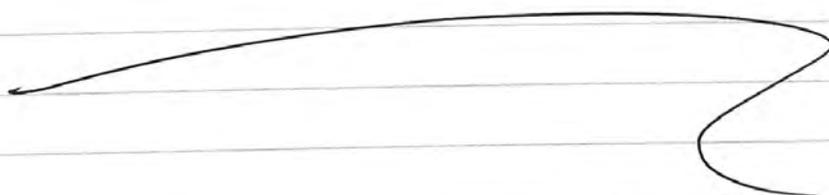
$$I_0 = \int_0^{\ln 2} [1] \, dx = [x]_0^{\ln 2} = \ln 2 //$$

$$\therefore I_2 = \frac{1}{2} \ln 2 + \frac{15}{32}$$

$$\therefore I_4 = \frac{3}{4} \left(\frac{1}{2} \ln 2 + \frac{15}{32} \right) + \frac{375}{1024}$$

$$I_4 = \frac{3}{8} \ln 2 + \frac{45}{128} + \frac{375}{1024}$$

$$I_4 = \frac{3}{8} \ln 2 + \frac{735}{1024}$$



$$(Q8a) \quad y = \ln \left(\frac{e^x + 1}{e^x - 1} \right)$$

$$\text{let } t = \frac{e^x + 1}{e^x - 1} = (e^x + 1)(e^x - 1)^{-1}$$

$$\text{then } \frac{dt}{dx} = - (e^x + 1)(e^x - 1)^{-2} (e^x)$$

$$+ (e^x)(e^x - 1)^{-1}$$

$$= (e^x - 1)^{-2} [-e^x(e^x + 1) + e^x(e^x - 1)]$$

$$= (e^x - 1)^{-2} [-e^{2x} - e^x + e^{2x} - e^x]$$

$$= (e^x - 1)^{-2} [-2e^x] = \frac{-2e^x}{(e^x - 1)^2} = \frac{dt}{dx}$$

$$y = \ln(t)$$

$$\frac{dy}{dt} = \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{1}{t} \times \frac{-2e^x}{(e^x - 1)^2}$$

$$= \left(\frac{e^x + 1}{e^x - 1} \right) \left(\frac{-2e^x}{(e^x - 1)^2} \right)$$

$$= \frac{-2e^x}{(e^x + 1)(e^x - 1)} = \boxed{\frac{-2e^x}{e^{2x} - 1}}$$

□

$$b) \text{ length} = \int_{\ln 2}^{\ln 3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4e^{2x}}{(e^{2x}-1)^2} = \frac{(e^{2x}-1)^2 + 4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{4x} - 2e^{2x} + 1 + 4e^{2x}}{(e^{2x}-1)^2}$$

$$= \frac{e^{4x} + 2e^{2x} + 1}{(e^{2x}-1)^2}$$

$$= \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2} =$$

$$\therefore \text{ length} = \int_{\ln 2}^{\ln 3} \sqrt{\frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}} dx = \int_{\ln 2}^{\ln 3} \left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx$$

$$\frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{\frac{e^x - e^{-x}}{x}} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} //$$

$$\text{hence length} = \int_{\ln 2}^{\ln 3} [\coth x] dx = \left[\ln \sinh x \right]_{\ln 2}^{\ln 3}$$

$$= \left[\ln \sinh(\ln 3) \right] - \left[\ln \sinh(\ln 2) \right]$$

$$= \ln \frac{4}{3} - \ln \frac{3}{4} = \boxed{\ln \frac{16}{9}}$$