

FP3 June 2018 (MA)

$$(Q1a) \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\frac{\sinh x}{\cosh x} = \tanh x = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$\downarrow \times e^x$
 $= \frac{e^{2x} - 1}{e^{2x} + 1}$

b) let $y = \operatorname{arctanh} \theta$,

$$\theta = \tanh(y)$$

$$\theta = \frac{e^{2y} - 1}{e^{2y} + 1}$$

$$\theta e^{2y} + \theta = e^{2y} - 1$$

$$\theta e^{2y} - e^{2y} + \theta + 1 = 0$$

$$e^{2y}(\theta - 1) = -1 - \theta$$

$$e^{2y} = \frac{-1 - \theta}{\theta - 1} = \frac{1 + \theta}{1 - \theta} //$$

$$\therefore 2y = \ln \left(\frac{1+\theta}{1-\theta} \right)$$

$$y = \frac{1}{2} \ln \left(\frac{1+\theta}{1-\theta} \right) //$$

$$\text{hence } \operatorname{artanh} \theta = \frac{1}{2} \ln \left(\frac{1+\theta}{1-\theta} \right)$$

(Q2a)

$$y = 5 \cosh x - 6 \sinh x = 0$$

$$5 \cosh x = 6 \sinh x$$

$$\div \cosh x : 5 = 6 \tanh x$$

$$\tanh x = \frac{5}{6} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$x = \operatorname{artanh} \left(\frac{5}{6} \right) = \frac{1}{2} \ln \left(\frac{1 + \frac{5}{6}}{1 - \frac{5}{6}} \right)$$

$$= \frac{1}{2} \ln \left(\frac{11/6}{1/6} \right) = \frac{1}{2} \ln 11$$

$$= \boxed{\ln \sqrt{11}}$$

$$\bullet \text{ Q3a) } M - \lambda I = \begin{pmatrix} 3-\lambda & u & 2 \\ -1 & -\lambda & 1 \\ 1 & u & 1-\lambda \end{pmatrix}$$

$$\det(M - \lambda I) = 3 - \lambda \begin{vmatrix} -\lambda & 1 \\ u & 1-\lambda \end{vmatrix} - u \begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} -1 & -\lambda \\ 1 & u \end{vmatrix} = 0$$

$$= (3 - \lambda) [-\lambda + \lambda^2 - u] - u [\lambda - 1 - 1] + 2 [\lambda - u] = 0$$

$$= -3\lambda + 3\lambda^2 - 3u + \lambda^2 - \lambda^3 + \lambda u - u\lambda + 2u + 2\lambda - 2u$$

$$= -\lambda^3 + 4\lambda^2 + \lambda(-3 + u - u + 2) - 3u$$

$$= -\lambda^3 + 4\lambda^2 - \lambda - 3u = 0 //$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + \lambda + 3u = 0$$

$\lambda - 3$ is a factor,

$$\begin{array}{r} \lambda^2 - \lambda - 2 \\ \lambda - 3 \overline{) \lambda^3 - 4\lambda^2 + \lambda + 3u} \\ \underline{\lambda^3 - 3\lambda^2} \\ 0 - \lambda^2 + \lambda \\ \underline{-\lambda^2 + 3\lambda} \\ 0 - 2\lambda + 3u \\ \underline{-2\lambda + 6} \\ 0 \quad 0 \end{array}$$

$$3u - 6 = 0$$

$$u = 2$$

remainder
should be 0

$$b) \Rightarrow (\lambda - 3)(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow (\lambda - 3)(\lambda - 2)(\lambda + 1) = 0$$

$$\boxed{\begin{array}{l} \lambda = 2 \\ \lambda = -1 \end{array}}$$

$$c) Mx = \lambda x$$

$$\begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$3x + 2y + 2z = 3x \quad \text{--- (1)}$$

$$\therefore y + z = 0 //$$

$$-x + z = 3y \quad \text{--- (2)}$$

$$x + 2y + z = 3z \quad \text{--- (3)}$$

$$\textcircled{1}: \text{ let } y = 1, z = -1 //$$

$$\hookrightarrow \textcircled{2}: x = z - 3y = -1 - 3(1) \\ = -4 //$$

So a corresponding eigenvector is...

$$\begin{pmatrix} -4 \\ 1 \\ -1 \end{pmatrix}$$

(Q4a) $y = \operatorname{arsinh} x + x(x^2+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}} + (x^2+1)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}(2x)$$

$$= \frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{1+x^2+x^2+1}{\sqrt{x^2+1}} = \frac{2(x^2+1)}{\sqrt{x^2+1}}$$

$$= \boxed{2\sqrt{x^2+1}}$$

b) length = $\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4(x^2+1) = 4x^2 + 5 //$$

$$\therefore \text{length} = \int_0^1 \sqrt{4x^2+5} dx$$

$$c) \int_0^1 \sqrt{4x^2 + 5} \, dx$$

⇓

$$= \int_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}} \left(\sqrt{4 \left(\frac{5}{4} \sinh^2 u \right) + 5} \right) \times \frac{\sqrt{5}}{2} \cosh u \, du$$

$$= \frac{\sqrt{5}}{2} \int_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}} \sqrt{5(1 + \sinh^2 u)} \times \cosh u \, du$$

$$= \frac{\sqrt{5}}{2} \int_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}} \sqrt{5} \times \sqrt{\cosh^2 u} \times \cosh u \, du = \frac{5}{2} \int_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}} (\cosh^2 u) \, du$$

$\cosh^2 u = 1 + \sinh^2 u$

$$\boxed{\begin{aligned} \cosh 2u &= 2\cosh^2 u - 1 \\ \frac{\cosh 2u + 1}{2} &= \cosh^2 u \end{aligned}}$$

$$\Rightarrow \frac{5}{4} \int_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}} (1 + \cosh 2u) \, du = \frac{5}{4} \left[u + \frac{1}{2} \sinh 2u \right]_0^{\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}}}$$

$$= \frac{5}{4} \left[\frac{as \frac{2}{\sqrt{5}}}{\sqrt{5}} + \frac{1}{2} \sinh \left(2as \frac{2}{\sqrt{5}} \right) \right]$$

$$x = \frac{\sqrt{5}}{2} \sinh u$$

$$\frac{dx}{du} = \frac{\sqrt{5}}{2} \cosh u$$

$$dx = \frac{\sqrt{5}}{2} \cosh u \, du$$

x	u
0	0
1	$\operatorname{arsinh} \left(\frac{2}{\sqrt{5}} \right)$

$$\operatorname{arsinh}\left(\frac{2}{\sqrt{5}}\right) = \ln\left(\frac{2}{\sqrt{5}} + \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + 1}\right)$$

$$= \ln(\sqrt{5})$$

$$\Rightarrow \frac{5}{4} \left[\ln(\sqrt{5}) + \frac{1}{2} \sinh(\ln 5) \right]$$

$$\Rightarrow \frac{5}{8} \times 2 \ln \sqrt{5} + \frac{1}{2} \left(\frac{12}{5} \right)$$

$$\Rightarrow \frac{5}{8} \ln 5 + \frac{12}{10} = \boxed{\frac{5}{8} \ln 5 + \frac{6}{5}}$$

(Q5a) $I_n = \int x^n (x+8)^{1/2} dx$

↳ By Parts

$$\frac{dv}{dx} = (x+8)^{1/2} \quad v = \frac{2}{3} (x+8)^{3/2}$$

$$u = x^n \quad u' = nx^{n-1}$$

$$\Rightarrow \left[\frac{2}{3} x^n (x+8)^{3/2} \right] - \frac{2}{3} n \int [x^{n-1} (x+8)^{3/2}] dx$$

$$x^{n-1} (x+8)^{3/2} = x^{n-1} (x+8)(x+8)^{1/2}$$

$$= x^n (x+8)^{1/2} + 8x^{n-1} (x+8)^{1/2}$$

$$\bullet \text{ (Q5a) } I_n = \int x^n (x+8)^{1/2} dx$$

$$\text{By Parts: } \frac{dv}{dx} = (x+8)^{1/2} \rightarrow v = \frac{2}{3} (x+8)^{3/2}$$

$$u = x^n \rightarrow n x^{n-1} = u'$$

$$\Rightarrow I_n = \left[\frac{2}{3} x^n (x+8)^{3/2} \right] - \frac{2}{3} n \int [x^{n-1} (x+8)^{3/2}] dx$$

$$\int x^{n-1} (x+8)^{3/2} dx = \int x^{n-1} (x+8) \sqrt{x+8} dx$$

$$= \int x^n \sqrt{x+8} dx + 8 \int x^{n-1} \sqrt{x+8} dx$$

$$\therefore I_n = \frac{2}{3} x^n (x+8)^{3/2} - \frac{2}{3} n \left[\int x^n \sqrt{x+8} dx + 8 \int x^{n-1} \sqrt{x+8} dx \right]$$

$$I_n = \frac{2}{3} x^n (x+8)^{3/2} - \frac{2}{3} n [I_n + 8 I_{n-1}]$$

$$3 I_n = 2 x^n (x+8)^{3/2} - 2n [I_n] - 2n [8 I_{n-1}]$$

$$(3+2n) I_n = 2 x^n (x+8)^{3/2} - 16n I_{n-1}$$

$$I_n = \frac{2 x^n (x+8)^{3/2}}{3+2n} - \frac{16n}{3+2n} I_{n-1}$$

$$b) I_2 = \left[\frac{2x^2(x+8)^{3/2}}{3+4} \right]_0^{10} - \frac{16(2)}{7} I_1,$$

$$= \frac{200}{7} (18)^{3/2} - \frac{32}{7} I_1,$$

$$= \frac{10800}{7} \sqrt{2} - \frac{32}{7} I_1,$$

$$I_1 = \left[\frac{2}{5} x(x+8)^{3/2} \right]_0^{10} - \frac{16}{5} I_0.$$

$$= \frac{20}{5} (10+8)^{3/2} - \frac{16}{5} I_0.$$

$$= 4(54\sqrt{2}) - \frac{16}{5} I_0.$$

$$= 216\sqrt{2} - \frac{16}{5} I_0.$$

$$I_0 = \int_0^{10} (x+8)^{1/2} dx = \left[\frac{2}{3} (x+8)^{3/2} \right]_0^{10}$$

$$= \frac{2}{3} (54\sqrt{2}) - \frac{2}{3} (16\sqrt{2}) = \frac{76}{3} \sqrt{2}$$

$$\begin{aligned}
 \therefore I_2 &= \frac{10800}{7} \sqrt{2} - \frac{32}{7} \left[216\sqrt{2} - \frac{16}{5} \left(\frac{76}{3} \sqrt{2} \right) \right] \\
 &= \frac{10800}{7} \sqrt{2} - \frac{6912}{7} \sqrt{2} + \frac{38912}{105} \sqrt{2} \\
 &= \left[\frac{(10800 - 6912)}{7} + \frac{38912}{105} \right] \sqrt{2} \\
 &= \boxed{\frac{97232}{105} \sqrt{2}}
 \end{aligned}$$

$$Q6a) l_2: r = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$l_1: r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

assume intersection:

$$\begin{pmatrix} -1 + \mu \\ 4 + \mu \\ 1 + 3\mu \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda \\ 0 + 3\lambda \\ 2 - \lambda \end{pmatrix} \sim \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}$$

$$\begin{aligned}
 \textcircled{2}: \mu = 3\lambda - 4 &\implies \textcircled{1}: -1 + 3\lambda - 4 = 1 + 2\lambda \\
 &\implies \lambda = 6 \\
 \textcircled{2}: \mu = 3(6) - 4 &= 14
 \end{aligned}$$

put $\lambda = 6$ into (3): $\mu = \frac{2 - 6 - 1}{3} = -\frac{5}{3}$

So values of μ and λ are not consistent so l_1 and l_2 don't intersect.

looking at direction vectors,

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \neq a \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

where a is an arbitrary constant.

so l_1 and l_2 aren't parallel.

Since l_1 and l_2 don't intersect and are not parallel then we can conclude they are indeed skew.

b) Shortest distance between 2 skew lines

$$= \left| \frac{(a-c) \cdot (b \times d)}{|b \times d|} \right|$$

$$(a-c) = \begin{pmatrix} -1 & -1 \\ 4 & -0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

$$(b \times d) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}$$

$$(a \cdot c) \cdot (b \times d) = \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 7 \\ 1 \end{pmatrix}$$

$$= 20 + 28 - 1 = 47 //$$

$$|b \times d| = \sqrt{10^2 + 7^2 + 1^2} = 5\sqrt{6}$$

$$\therefore \text{distance} = \frac{47}{5\sqrt{6}} = \boxed{\frac{47\sqrt{6}}{30}}$$

c) Π contains the points $\begin{pmatrix} 3 \\ 8 \\ 13 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
and the line L .

We must find a normal to Π .

$$\vec{AB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \\ 13 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ -11 \end{pmatrix}$$

$[\vec{AB}] \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ will yield a normal to Π

$$\begin{pmatrix} -2 \\ -8 \\ -11 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} //$$

So... $r \cdot \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} = \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 13 \end{pmatrix} = 4(13) - 24(8) + 136$

$$r \cdot \begin{pmatrix} 41 \\ -24 \\ 10 \end{pmatrix} = 61$$

So cartesian eqn :

$$41x - 24y + 10z = 61$$

Q7a) $ae = 3 \quad \frac{a}{e} = \frac{25}{3}$

$$\text{so } a = \frac{25e}{3} //$$

$$\therefore \frac{25}{3}e^2 = 3$$

$$b^2 = a^2(1 - e^2) \quad \left\{ \begin{array}{l} e^2 = \frac{9}{25} \quad \text{so } e = \boxed{\frac{3}{5}} // \\ \therefore a = \frac{25}{3} \times \frac{3}{5} = \boxed{5} // \end{array} \right.$$

$$b^2 = 25\left(1 - \frac{9}{25}\right)$$

$$b^2 = 16$$

$$\therefore \boxed{\frac{x^2}{25} + \frac{y^2}{16} = 1} //$$

$$b) \quad \frac{x^2}{25} + \frac{(mx+c)^2}{16} = 1$$

$$\frac{x^2}{25} + \frac{m^2x^2 + 2mxc + c^2}{16} = 1$$

$$\times 400 : 16x^2 + 25m^2x^2 + 50mxc + 25c^2 = 400$$

$$(16 + 25m^2)x^2 + (50mc)x + 25c^2 - 400 = 0$$

$$\text{hence } (25m^2 + 16)x^2 + (50mc)x + 25(c^2 - 16) = 0$$

$$c) \quad l \text{ is a tangent to } E \quad \therefore b^2 - 4ac = 0$$

$$\left. \begin{array}{l} a = 16 + 25m^2 \\ b = 50mc \\ c = 25(c^2 - 16) \end{array} \right\} \quad \begin{array}{l} b^2 - 4ac = 0 \\ (50mc)^2 - 4(16 + 25m^2)(25)(c^2 - 16) = 0 \end{array}$$

$$\Rightarrow 2500m^2c^2 - 100(16c^2 - 256 + 25m^2c^2 - 400m^2) = 0$$

$$\Rightarrow 2500m^2c^2 - 1600c^2 + 25600 - 2500m^2c^2 + 40000m^2 = 0$$

$$\Rightarrow -1600c^2 + 25600 + 40000m^2 = 0$$

$$\div -1600 : \quad c^2 - 16 - 25m^2 = 0$$

$$\therefore \boxed{c^2 = 25m^2 + 16}$$

$$\bullet \text{ d) Area OAB} = \left| \frac{1}{2} (OA)(OB) \right|$$

$$y = mx + c \quad \text{and} \quad c^2 = 25m^2 + 16$$

$$\underline{x=0} : y = c$$

$$\underline{y=0} : x = -\frac{c}{m}$$

$$\therefore \text{Area} = \left| \frac{1}{2} (c) \left(-\frac{c}{m} \right) \right| = \left| \frac{-c^2}{2m} \right|$$

$$= \frac{c^2}{2m} = \frac{25m^2 + 16}{2m}$$

$$\bullet \text{ e) } A = \frac{25m^2}{2m} + \frac{16}{2m} = \frac{25m}{2} + 8m^{-1}$$

$$\therefore \frac{dA}{dm} = \frac{25}{2} - 8m^{-2} \quad // \quad = 0$$

$$\frac{25}{2} = \frac{8}{m^2}$$

$$\frac{2}{25} = \frac{m^2}{8}$$

$$\therefore m^2 = \frac{16}{25} \rightarrow m = \frac{4}{5} \quad (m > 0)$$

$$\text{So } A_{\min} = \frac{25 \left(\frac{16}{25} \right) + 16}{2 \left(\frac{4}{5} \right)}$$

$$= \frac{16 + 16}{\frac{8}{5}} = \boxed{20} \text{ units}^2$$