

M3 January 2014 IAL (MA)

$$Q1) \quad v = \sqrt{8x^{\frac{3}{2}} - 4} = (8x^{\frac{3}{2}} - 4)^{\frac{1}{2}}$$

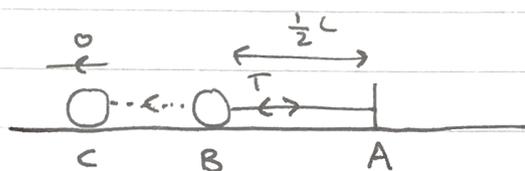
$$\frac{dv}{dx} = \frac{1}{2} (8x^{\frac{3}{2}} - 4)^{-\frac{1}{2}} (12x^{\frac{1}{2}}) = \frac{6\sqrt{x}}{\sqrt{8x^{\frac{3}{2}} - 4}} //$$

$$a = v \frac{dv}{dx} = \frac{\sqrt{8x^{\frac{3}{2}} - 4}}{\sqrt{8x^{\frac{3}{2}} - 4}} \times \frac{6\sqrt{x}}{\sqrt{8x^{\frac{3}{2}} - 4}} = 6\sqrt{x} //$$

$$\Sigma F = ma : \quad F = 3\sqrt{x}.$$

$$\text{at } x=4, \quad F = 3\sqrt{4} = \boxed{6 \text{ N}}$$

Q2)



$$\text{At B : } KE = 0$$

$$EPE = \frac{2mg\left(\frac{l}{2}\right)^2}{2l} = \frac{mg l^2}{4l} = \frac{mgl}{4} //$$

$$\text{At C : } KE = 0$$

$$EPE = \frac{2mg\left(d - \frac{l}{2}\right)^2}{2l}$$

$$+ \text{W.D due to friction} = \frac{1}{4}(mg)d.$$

$$\text{So by C.O.E : } \frac{mgl}{4} = \frac{mg}{l} \left(d - \frac{l}{2}\right)^2 + \frac{mgd}{4}$$

$$\frac{L}{4} = \frac{1}{L} \left(d - \frac{L}{2} \right)^2 + \frac{d}{4}$$

$$\frac{L}{4} = \frac{1}{L} \left(d^2 - dL + \frac{L^2}{4} \right) + \frac{d}{4}$$

$$\times L: \quad \cancel{\frac{L^2}{4}} = d^2 - dL + \cancel{\frac{L^2}{4}} + \frac{dL}{4}$$

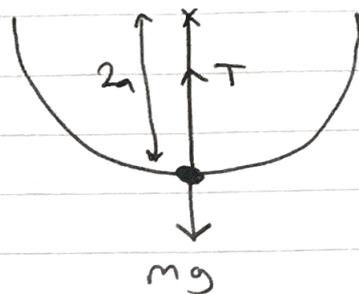
$$\Rightarrow d^2 - dL + \frac{dL}{4} = 0$$

$$\Rightarrow d \left(d - L + \frac{L}{4} \right) = 0$$

$$\Rightarrow d - L + \frac{L}{4} = 0$$

$$\Rightarrow \boxed{d = \frac{3L}{4}} = \text{distance from B.}$$

Q3) Greatest tension is at the lowest point.



$$\text{NZL } \uparrow (P): T - mg = \frac{mv^2}{r}$$

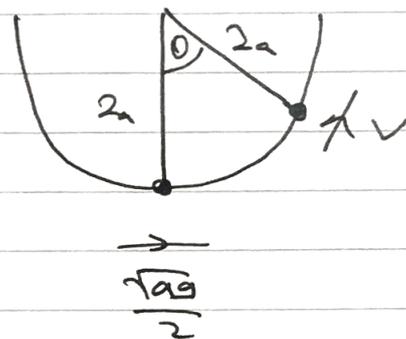
$$\frac{9mg}{8} - mg = \frac{mv^2}{2a}$$

$$\frac{g}{8}(2a) = v^2 = \frac{ag}{4} //$$

Using C.O.E from lowest point to when P makes an angle θ with downward vertical,

Initially: $KE = \frac{1}{2} m \left(\frac{ag}{4} \right) = \frac{amg}{8}$

$$GPE = 0$$



Finally: $KE = \frac{1}{2} m \left(\frac{ag}{20} \right)$

$$GPE = mg(2a - 2a \cos \theta)$$

$$\Rightarrow \frac{amg}{8} = \frac{amg}{40} + 2amg(1 - \cos \theta)$$

$$\Rightarrow \frac{\frac{1}{8} - \frac{1}{40}}{2} = 1 - \cos \theta = \frac{1}{20}$$

$$\therefore \cos \theta = \frac{19}{20} //$$

$$\theta = \cos^{-1} \frac{19}{20} = \boxed{18^\circ}$$

$$\begin{aligned}
 \text{Q4a)} \quad V &= \pi \int_0^1 y^2 dx = \pi \int_0^1 [e^{-2x}] dx \\
 &= \left[-\frac{1}{2} \pi e^{-2x} \right]_0^1 = \left[-\frac{1}{2} \pi e^{-2} \right] + \left[\frac{1}{2} \pi \right] \\
 &= \frac{\pi}{2} (1 - e^{-2})
 \end{aligned}$$

$$\text{b)} \quad M\bar{x} = \pi \int_0^1 y^2 x dx = \pi \int_0^1 [x e^{-2x}] dx$$

By Parts : $\frac{dv}{dx} = e^{-2x} \rightarrow v = -\frac{1}{2} e^{-2x}$

$$u = x \rightarrow u' = 1$$

$$\Rightarrow \left[-\frac{1}{2} x e^{-2x} \right]_0^1 + \frac{1}{2} \int_0^1 (e^{-2x}) dx$$

$$\Rightarrow \left[-\frac{1}{2} e^{-2} \right] - \left[\frac{1}{4} e^{-2x} \right]_0^1$$

$$\Rightarrow -\frac{1}{2} e^{-2} - \left[\frac{1}{4} e^{-2} - \frac{1}{4} \right]$$

$$\Rightarrow -\frac{3}{4} e^{-2} + \frac{1}{4}$$

$$\therefore M\bar{x} = \pi \left[\frac{1}{4} - \frac{3}{4} e^{-2} \right]$$

and from (a), $M = \frac{\pi}{2} (1 - e^{-2}) //$

$$\text{so } \bar{x} = \frac{\pi \left[\frac{1}{4} - \frac{3}{4} e^{-2} \right]}{\frac{\pi}{2} (1 - e^{-2})} = \frac{\frac{1}{2} - \frac{3}{2} e^{-2}}{1 - e^{-2}}$$

$$= \frac{1 - 3e^{-2} \quad (\times e^2)}{2 - 2e^{-2} \quad (\times e^2)} = \frac{e^2 - 3}{2e^2 - 2}$$

$$= \frac{e^2 - 3}{2(e^2 - 1)}$$

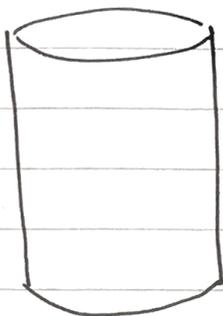
Q5a) Shape Mass (vol.) Distance of c.o.m from O



$$\frac{2}{3}\pi r^3(3\mu)$$

$$= \boxed{2\mu\pi r^3}$$

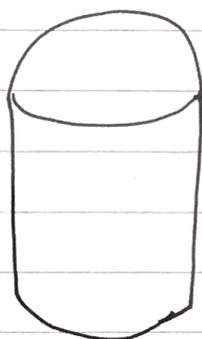
$$\frac{3r}{8} + 3r = \boxed{\frac{27r}{8}}$$



$$\pi r^2(3r)(\mu)$$

$$= \boxed{3\pi r^3\mu}$$

$$\boxed{\frac{3r}{2}}$$



$$\boxed{5\pi r^3\mu}$$

$$\boxed{\bar{y}}$$

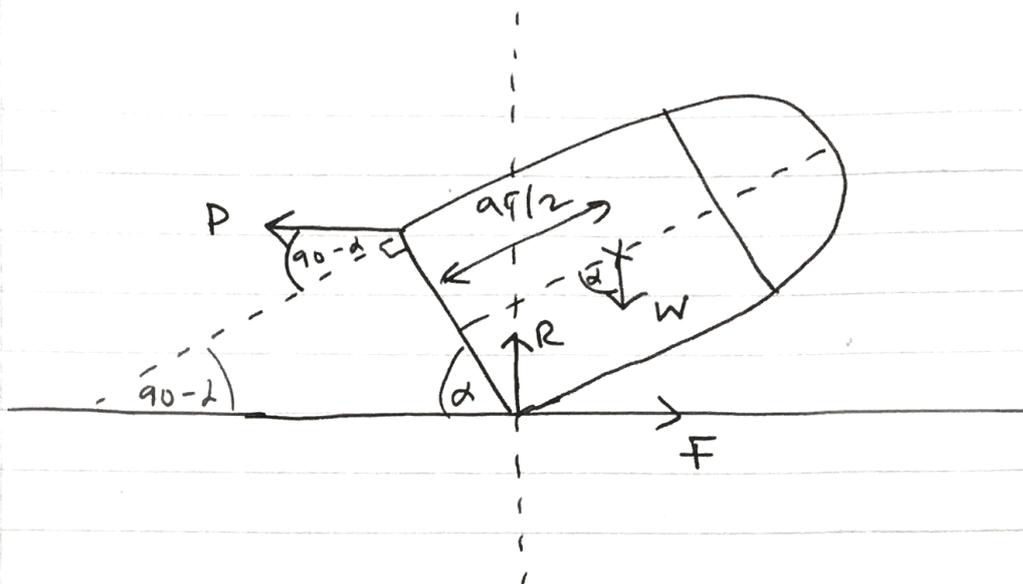
taking moments about diameter through O,

$$2\pi r^3\mu \left(\frac{27r}{8}\right) + 3\pi r^3\mu \left(\frac{3r}{2}\right) = 5\pi r^3\mu (\bar{y})$$

$$\frac{27r}{4} + \frac{9r}{2} = 5\bar{y} = \frac{45r}{4}$$

$$\therefore \bar{y} = \frac{45r}{5 \times 4} = \boxed{\frac{9r}{4}}$$

5b)



$$\left. \begin{array}{l} R(\updownarrow): W = R \\ R(\leftrightarrow): F = P \end{array} \right\} \begin{array}{l} F = \mu R \\ P = \mu W. \end{array}$$

$$M(\text{Point of contact}) : W \left[\sin \alpha \left(\frac{9r}{4} \right) - r \cos \alpha \right] = P (\cos(90 - \alpha)) \times 2r$$

$$\Rightarrow W \left(\frac{9r}{4} \sin \alpha - r \cos \alpha \right) = 2P \sin \alpha$$

$$\div \sin \alpha : W \left(\frac{9r}{4} - \cot \alpha \right) = 2P$$

$$\therefore P = \frac{W}{2} \left(\frac{9r}{4} - \cot \alpha \right)$$

$$\text{and } P = \mu W$$

$$\Rightarrow \frac{W}{2} \left(\frac{9r}{4} - \cot \alpha \right) = \mu W$$

$$\Rightarrow \mu = \frac{9r}{8} - \frac{1}{2} \cot \alpha = \frac{1}{8} (9r - 4 \cot \alpha)$$

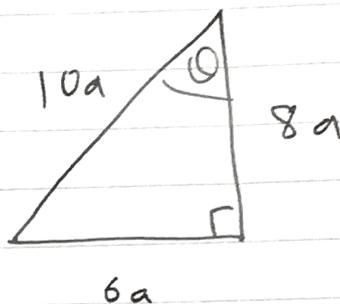
REMEMBER: We are taking moments about the dotted line through the point of contact. So find perpendicular distances from the lines of action.

of P/F to the dotted line.

(Q6a) $\sqrt{(10a)^2 - (8a)^2} = 6a //$

hence $\angle APB = 90^\circ$

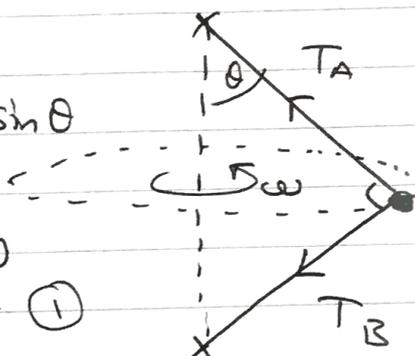
(Pythagoras)



b) $R(\downarrow) : T_A \cos \theta = mg + T_B \sin \theta$

$\therefore T_A \cos \theta - T_B \sin \theta = mg$

①



$\cos \theta = \frac{8}{10}$

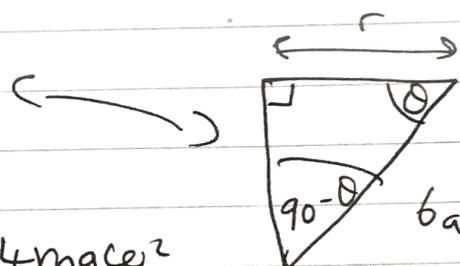
$\sin \theta = \frac{6}{10}$

$\tan \theta = \frac{3}{4}$

$\leftarrow N2L(P) : T_A \sin \theta + T_B \cos \theta = mr\omega^2$

$r = 6a \cos \theta$

$r = 6a \times \frac{4}{5} = \frac{24a}{5} //$



② $\therefore T_A \sin \theta + T_B \cos \theta = \frac{24ma\omega^2}{5}$

Now remember that the lower part of the string must remain taut for P to continue moving in circular motion.

so $T_B \geq 0$.

solve ① and ② for T_B :

$$\textcircled{1} \div \cos \theta : T_A - T_B \tan \theta = \frac{5mg}{4}$$

$$T_A - \frac{3}{4} T_B = \frac{5mg}{4}$$

$$\textcircled{2} \div \cos \theta : T_A \left(\frac{3}{4}\right) + T_B = 6mace^2$$

$\times 4/3$

$$\Rightarrow T_A + \frac{4}{3} T_B = 8mace^2$$

$$\Rightarrow T_A = 8mace^2 - \frac{4}{3} T_B$$

$$\text{from } \textcircled{1} : T_A = \frac{3}{4} T_B + \frac{5mg}{4}$$

$$\textcircled{1} = \textcircled{2} : \frac{3T_B}{4} = \frac{4}{3} T_B - \frac{5mg}{4} + 8mace^2$$

$$\frac{25}{12} T_B = 8mace^2 - \frac{5mg}{4}$$

$$\frac{25}{12} T_B = \frac{m}{4} (32ace^2 - 5g)$$

$$\therefore T_B = \frac{3m}{25} (32ace^2 - 5g)$$

$$T_B > 0$$

$$32ace^2 - 5g \geq 0$$

$$\omega^2 \geq \frac{5g}{32a} \quad \therefore \omega \geq \sqrt{\frac{5g}{32a}}$$

$$\text{and } T = \frac{2\pi}{\omega} \rightarrow \omega = \frac{2\pi}{T}$$

$$\therefore \sqrt{\frac{5g}{32a}} \leq \frac{2\pi}{T}$$

take reciprocal of both sides so signs flip...

$$T \leq \frac{2\pi}{\sqrt{\frac{5g}{32a}}} \rightarrow T \leq 2\pi \sqrt{\frac{32a}{5g}}$$

(max time is $2\pi \sqrt{\frac{32a}{5g}}$)

Q7a) $T = mg$



$$\frac{8mg e}{L} = mg$$

~~L~~

$$\therefore 8e = L \quad \therefore \boxed{\frac{L}{8} = e}$$

b) NZUP \uparrow^+ : $-mg - T = m\ddot{x}$

$T = \frac{\lambda x}{L} = \frac{8mg}{L} \left(x - \frac{L}{8}\right)$

where x is distance from equilibrium position,

$$\therefore -mg - \frac{8mg}{L} \left(x - \frac{L}{8}\right) = m\ddot{x}$$

$$-mg - \frac{8mgx}{L} + mg = m\ddot{x}$$

$$\therefore -\frac{8gx}{L} = \ddot{x}$$

so $\ddot{x} = -\left(\frac{8g}{L}\right)x$ hence P moves with S.H.M.

$$\omega^2 = \frac{8g}{L} \rightarrow \omega = \sqrt{\frac{8g}{L}}$$

$$T = 2\pi \times \frac{1}{\omega} = 2\pi \times \sqrt{\frac{L}{8g}}$$

c) Initially :

$$KE = 0$$

$$GPE = 0$$

$$EPE = \frac{8mg}{2L} \left(\frac{L}{2}\right)^2$$

Finally :

$$KE = \frac{1}{2}mv^2$$

$$GPE = mg\left(\frac{L}{2}\right)$$

$$EPE = \frac{8mg}{2L}(0) = 0 //$$

$$\Rightarrow \frac{8mg}{2L} \left(\frac{L^2}{4}\right) = \frac{mv^2}{2} + \frac{mgL}{2}$$

$$\Rightarrow \frac{4 \cdot gL^2}{4L} = \frac{v^2}{2} + \frac{gL}{2}$$

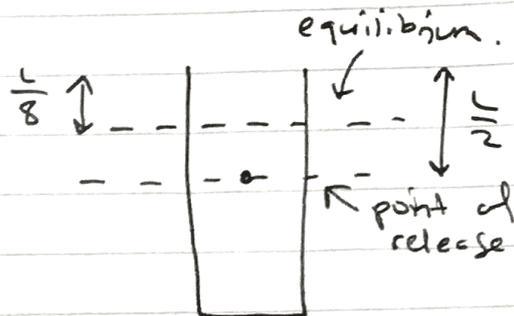
$$\Rightarrow gL - \frac{gL}{2} = \frac{v^2}{2} = \frac{gL}{2}$$

$$\therefore v^2 = gL \rightarrow \boxed{v = \sqrt{gL} = u}$$

d) the fact that P is released at an 'endpoint' means the characteristic S.H.M equation to use is: $x = a \cos \omega t$.

$$\dot{x} = -a\omega \sin \omega t$$

$$a = \frac{l}{2} - \frac{l}{8} = \frac{3l}{8}$$



$$\therefore \ddot{x} = -\frac{3l}{8} \sqrt{\frac{8g}{l}} \sin \left(t \sqrt{\frac{8g}{l}} \right)$$

$$-\sqrt{\frac{9g}{32}} = -\frac{3}{8} \sqrt{8g} \sin \left(t \sqrt{\frac{8g}{l}} \right)$$

$$\frac{-\sqrt{\frac{9}{32}}}{-\frac{3}{8}\sqrt{8}} = \sin \left(t \sqrt{\frac{8g}{l}} \right) = \frac{1}{2}$$

x will be negative as P will be below the equilibrium point when it first attains a speed of $\sqrt{\frac{9gl}{32}}$.

$$\therefore t \sqrt{\frac{8g}{l}} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6} \times \sqrt{\frac{l}{8g}} = t$$

