M3 June 2014 (R) (MA)

Q1a) $\frac{dv}{dx} = 3$

$\int (1)\, dv = 3\int (1)\, dx$

$v = 3x + c$

$v=3, x=2$: $3 = 6 + c$

$c = -3$

$\therefore v = 3x - 3$

at $x=5$, $v = 3(5) - 3 = 12 ms^{-1}$

$\therefore v\frac{dv}{dx} = a = 12(3) = 36 ms^{-2}$ at $x = 5$

$F = ma = 0.25 \times 36 = \boxed{9N}$

b) $v = 3x - 3$

$\frac{dx}{dt} = 3x - 3$

$\left(\frac{1}{x-1}\right)\frac{dx}{dt} = 3$

$\int \frac{1}{x-1}\, dx = \int 3\, dt$

$\therefore \ln(x-1) = 3t + c$

$x = 2,\ t = 0$: $\ln(1) = 0 + c$

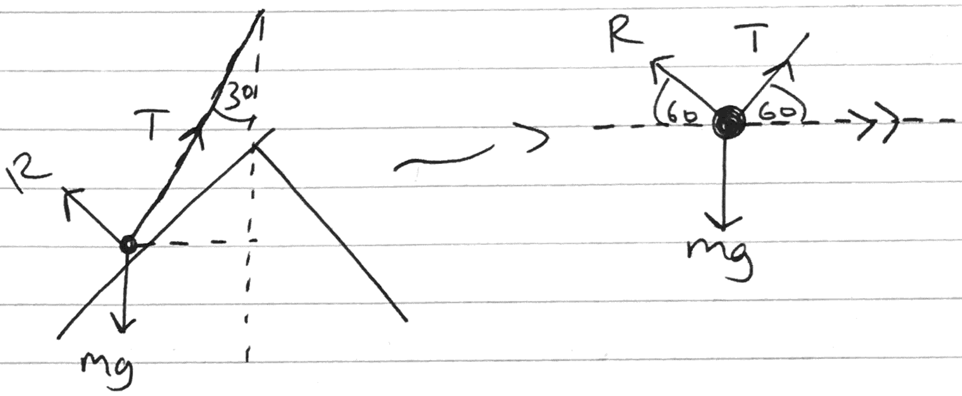
$c = 0$

so $\ln(x-1) = 3t$

$x = 5$: $\ln 4 = 3t$

$\therefore t = \frac{1}{3}\ln 4 \simeq 0.462$

(Q2a)



$\xrightarrow{\quad\quad} +$

r

N2L(particle) : $T\cos 60 - R\cos 60 = m(l\sin 30)\omega^2$

$\frac{T-R}{2} = \frac{ml}{2}\omega^2$

$\therefore T - R = ml\omega^2 \sim (1)$

$R(\updownarrow)$: $T\sin 60 + R\sin 60 = mg$

$T + R = \frac{mg}{\sin 60}$

$R = \frac{2mg\sqrt{3}}{3} - T$

sub R into ①: $T - \frac{2mg\sqrt{3}}{3} + T = ml\omega^2$

$$2T = m\left(l\omega^2 + \frac{2g\sqrt{3}}{3}\right)$$

$$\therefore T = \frac{m}{2}\left(l\omega^2 + \frac{2g\sqrt{3}}{3}\right)$$

b) particle remains on the surface so $R > 0$.

$$R = \frac{2mg\sqrt{3}}{3} - T$$

$$R = \frac{2mg\sqrt{3}}{3} - \frac{ml\omega^2}{2} - \frac{mg\sqrt{3}}{3}$$

$$R = \frac{mg\sqrt{3}}{3} - \frac{ml\omega^2}{2}$$

$R > 0$: $\frac{\cancel{m}g\sqrt{3}}{3} - \frac{\cancel{m}l\omega^2}{2} > 0$

$$\frac{l\omega^2}{2} < \frac{g\sqrt{3}}{3}$$

$$\omega^2 < \frac{2g\sqrt{3}}{3l}$$

$$\therefore \omega < \sqrt{\frac{2g\sqrt{3}}{3l}}.$$

$$T = \frac{2\pi}{\omega}$$

$$\frac{2\pi}{T} = \omega$$

so $\dfrac{2\pi}{T} < \sqrt{\dfrac{2g\sqrt{3}}{3l}}$

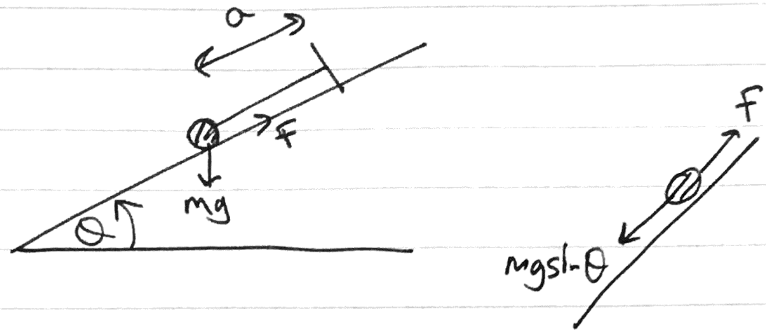
take reciprocal of both sides so signs flip:

$$\frac{T}{2\pi} > \sqrt{\frac{3l}{2g\sqrt{3}}}$$

either form is ok.

$$T > 2\pi\sqrt{\frac{3l}{2g\sqrt{3}}} \longrightarrow T > 2\pi\sqrt{\frac{l\sqrt{3}}{2g}}$$

Q3a)



Initially: GPE = $mgd\sin\theta$
KE = 0
EPE = 0

[let distance moved down the plane before coming to rest = d]

Finally: KE = 0
GPE = 0
EPE = $\frac{mg}{2a}(d)^2$

+ W·D by friction = $F \times d = \mu R \times d = (mg\cos\theta)d\cdot\mu$

C.O.E

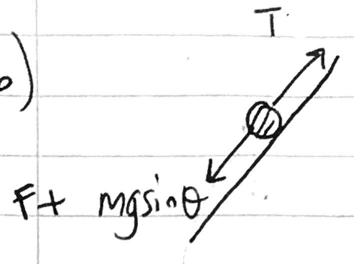
$$\Rightarrow \underbrace{mgd\sin\theta}_{\text{total energy initially}} = \underbrace{\frac{mgd^2}{2a}}_{\text{total energy finally}} + \underbrace{mgd\mu\cos\theta}_{\text{work done by friction}}$$

÷ mgd : $\sin\theta = \frac{d}{2a} + \mu\cos\theta$

× 2a : $2a\sin\theta - 2a\mu\cos\theta = d$

$\therefore d = 2a(\sin\theta - \mu\cos\theta)$

b)



$R(\nearrow)$: $F + mg\sin\theta = T$

$F + mg\sin\theta = \frac{mg}{a}(d)$

$F = \frac{mg}{a}(2a)(\sin\theta - \mu\cos\theta) - mg\sin\theta$

$F = 2mg\sin\theta - 2mg\mu\cos\theta - mg\sin\theta$

$F = mg\sin\theta - 2mg\mu\cos\theta$

but $F \leq \mu R$.

$R = mg\cos\theta$

$F \leq \mu mg\cos\theta$

and $F = mg\sin\theta - 2mg\mu\cos\theta$

$\therefore$ $mg\sin\theta - mg\mu\cos\theta \leq 2\mu mg\cos\theta$

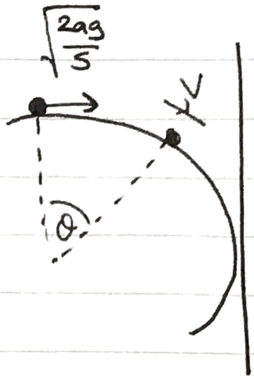
$mg\sin\theta \leq 3mg\mu\cos\theta$

$\sin\theta \leq 3\mu\cos\theta$

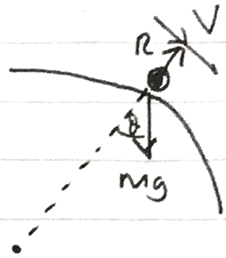
hence $\mu \geq \frac{\sin\theta}{3\cos\theta}$

$\Rightarrow$ $\mu \geq \frac{1}{3}\tan\theta$

(Q4a)



when the particle leaves the sphere



$\swarrow^{+}$ N2L (particle) : $mg\cos\theta - R = \frac{mv^2}{a}$

$\Rightarrow$ $R=0$: $\cancel{m}g\cos\theta = \frac{\cancel{m}v^2}{a}$

$\Rightarrow$ $v^2 = ag\cos\theta$.

Energy from top to when particle leaves sphere:

At top : $KE = \frac{1}{2}m\left(\frac{2ag}{5}\right) = \frac{2amg}{10} = \frac{amg}{5}$.

$GPE = mga$

At point where particle leaves the sphere : $KE = \frac{1}{2}mv^2$

$GPE = mga\cos\theta$

C.O.E : $\frac{a\cancel{m}g}{5} + a\cancel{m}g = \frac{\cancel{m}v^2}{2} + a\cancel{m}g\cos\theta$

$\frac{6}{5}ag - ag\cos\theta = \frac{v^2}{2}$

$v^2 = \frac{12}{5}ag - 2ag\cos\theta$

but $v^2 = ag\cos\theta$: $ag\cos\theta = \frac{12ag}{5} - 2ag\cos\theta$

$3\cancel{ag}\cos\theta = \frac{12\cancel{ag}}{5}$

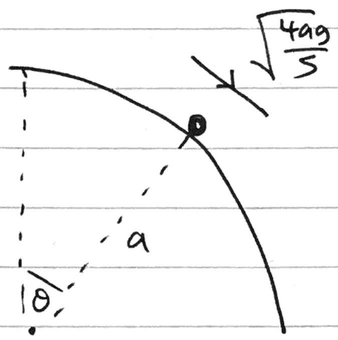
$$3\cos\theta = \frac{12}{5}$$

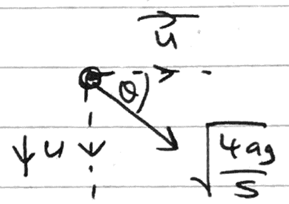
$$\therefore \cos\theta = \frac{12}{15} = \frac{4}{5}$$

$$\text{so } v^2 = ag\left(\frac{4}{5}\right)$$

$$\therefore v = \sqrt{\frac{4ag}{5}}$$

b)





$$\text{so } \vec{u} = \cos\theta\sqrt{\frac{4ag}{5}}$$

$$\vec{u} = \frac{4}{5}\sqrt{\frac{4ag}{5}}$$

$$\text{and } u\downarrow = \sqrt{\frac{4ag}{5}}\sin\theta = \frac{3}{5}\sqrt{\frac{4ag}{5}}$$

distance from wall initially $= a - a\sin\theta$

$$= a(1-\sin\theta)$$

$$= a\left(1-\frac{3}{5}\right) = \frac{2a}{5}$$

consider horizontal motion: $\vec{s} = ut$.

$$\Rightarrow \frac{2a}{5} = \left(\frac{4}{5}\sqrt{\frac{4ag}{5}}\right)t$$

$$2a = \left(4\sqrt{\frac{4ag}{5}}\right)t$$

$$\frac{a}{2\sqrt{\frac{4ag}{5}}} = t = \frac{a}{2}\sqrt{\frac{5}{4ag}} = \frac{1}{2}\sqrt{\frac{5a}{4g}}$$

consider vertical motion:

$s = x$
$u = \frac{3}{5}\sqrt{\frac{4ag}{5}}$
$v =$
$a = g$
$t = \frac{1}{2}\sqrt{\frac{5a}{4g}}$

$$s = ut + \tfrac{1}{2}at^2$$

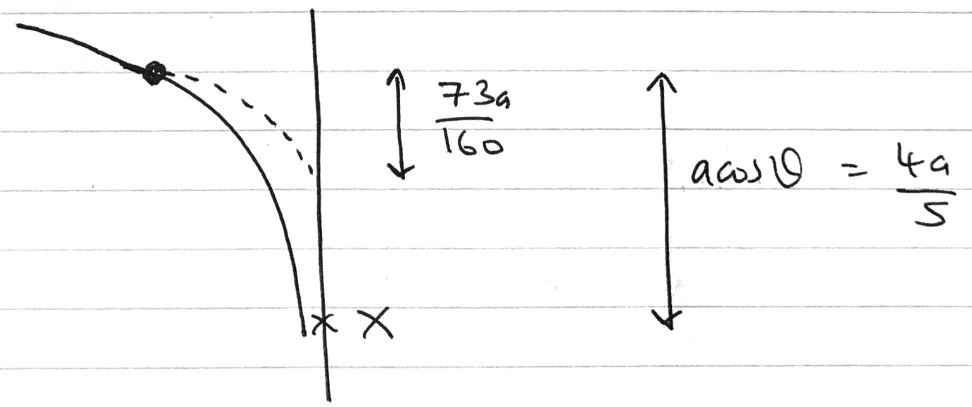
$$x = \frac{3}{10}\sqrt{\frac{4ag}{\cancel{5}}}\sqrt{\frac{\cancel{5}a}{\cancel{4}g}} + \frac{g}{2}\left(\frac{1}{4}\right)\left(\frac{5a}{4g}\right)$$

$$x = \frac{3}{10}\sqrt{a^2} + \frac{5a}{32}$$

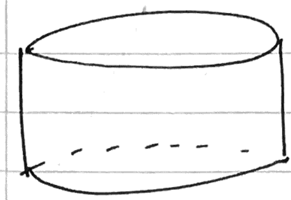
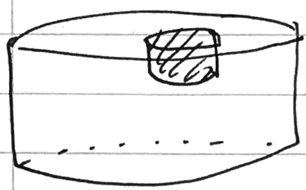
$$x = \frac{73}{160}a = \text{distance} \downarrow \text{ travelled}$$

$$\text{so } AX = \frac{4a}{5} - \frac{73}{160}a = \boxed{\frac{11a}{32}}$$

$a\cos\theta$



(Q5a)

Shape	Mass (vol.)	Distance of c.o.m from O
	$\boxed{\pi r^2 h}$	$\boxed{\frac{h}{2}}$
$-$ 	$\pi\left(\frac{1}{4}r\right)^2\left(\frac{h}{4}\right) = \boxed{\frac{\pi r^2 h}{64}}$	$\boxed{\frac{h}{8}}$
	$\boxed{\frac{63\pi r^2 h}{64}}$	$\boxed{\bar{y}}$

taking moments about a diameter of the plane face containing O...

$$\cancel{\pi r^2 h}\left(\frac{h}{2}\right) - \frac{\cancel{\pi r^2 h}}{64}\left(\frac{h}{8}\right) = \frac{63\cancel{\pi r^2 h}}{64}\left(\bar{y}\right)$$

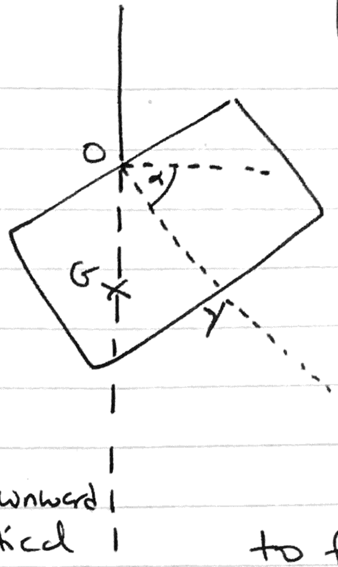
$$\frac{\frac{h}{2} - \frac{h}{8(64)}}{\frac{63}{64}} = \bar{y} = \frac{\frac{255h}{512}}{\frac{63}{64}}$$

$$= \frac{85h}{168}$$

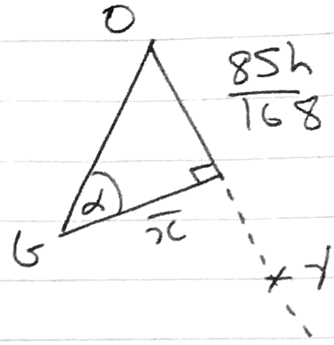
We are told that

$$\boxed{\tan\alpha = 17}$$

b)



Downward vertical



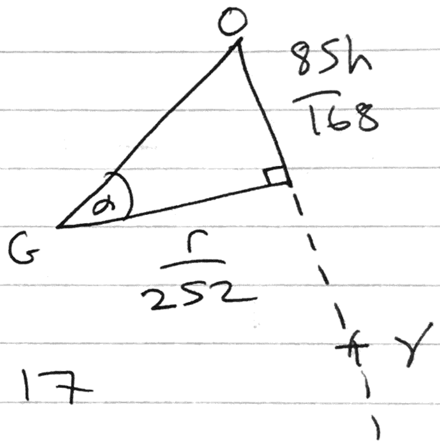
to find $\bar{x}$ now,

moments about O, but use displacement of c.o.m from OY this time

$$\cancel{\pi r^2 h}(0) - \frac{\pi r^2 h}{64}\left(\frac{r}{4}\right) = \frac{63}{64}\cancel{\pi r^2 h}(\bar{x})$$

$$\bar{x} = \frac{-\frac{r}{64(4)}}{\frac{63}{64}} = \frac{-r}{252}$$

So now:



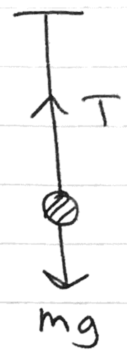
$$\tan\alpha = \frac{\frac{85h}{168}}{\frac{r}{252}} = 17$$

$$\frac{17r}{252} = \frac{85h}{168}$$

$$r = \frac{85}{168}h \times \frac{252}{17} = \frac{15h}{2}$$

$$\therefore \boxed{r = \frac{15h}{2}}$$

Q6a) $T = mg = \frac{\lambda x}{l}$



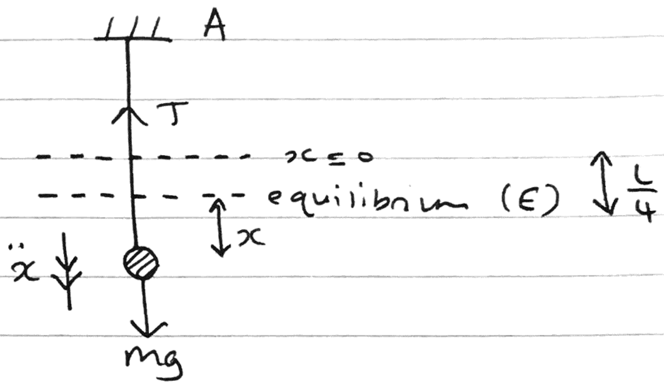
$$\frac{4mg}{l}(e) = mg$$

$\div mg$

$$\Rightarrow \frac{4e}{l} = 1$$

$$\Rightarrow \boxed{e = \frac{l}{4}}$$

b) $mg - T = m\ddot{x}$ [$\downarrow_{+}$ (N2L(P))]



$$T = \frac{4mg}{l}\left(\frac{l}{4} + x\right)$$

$$\therefore mg - \frac{4mgl}{4l} - \frac{4mgx}{l} = m\ddot{x}$$

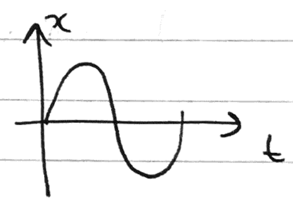
$$\Rightarrow mg - mg - \frac{4mgx}{l} = m\ddot{x}$$

$$\Rightarrow -\frac{4mgx}{l} = m\ddot{x}$$

$$\Rightarrow \ddot{x} = -\frac{4gx}{l}$$

$\therefore$ P moves with S.H.M.

P starts at equilibrium position so $x = a\sin\omega t$ applies.



So the initial velocity of P is positive (as seen from the graph). hence x increases in the direction AE. So $\ddot{x}$ also increases in this direction so $\ddot{x}$ is positive $\downarrow$.

c) $\ddot{x} = -\frac{4g}{l}x$

$$\omega = \sqrt{\frac{4g}{l}}$$

speed of projection = max speed
(as this is at the centre of the oscillation)

$$\therefore V_{max} = \sqrt{gl} = a\omega$$

$$\sqrt{gl} = a\sqrt{\frac{4g}{l}}$$

$$\sqrt{l} = a\sqrt{\frac{4}{l}}$$

$$l = a\sqrt{4}$$

$$\boxed{a = \frac{l}{2}}$$

d) When string first becomes slack, $x = -\frac{l}{4}$

($x = -\frac{l}{4}$ as above the equilibrium point x is negative)

$x = a\sin\omega t$:

$$-\frac{l}{4} = \frac{l}{2}\sin\left(t\sqrt{\frac{4g}{l}}\right)$$

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \frac{7\pi}{6}$

$$-\frac{1}{2} = \sin\left(t\sqrt{\frac{4g}{l}}\right)$$

$$t\sqrt{\frac{4g}{l}} = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \frac{7\pi}{6}$$

$t > 0 \quad \therefore \quad \frac{7\pi}{6} = t\sqrt{\frac{4g}{l}}$

$$\text{so } t = \frac{7\pi}{6}\sqrt{\frac{l}{4g}} = \frac{7\pi}{12}\sqrt{\frac{l}{2g}}$$