

M3 May 13 2015

1. A particle P of mass 0.5 kg is attached to one end of a light elastic spring, of natural length 1.2 m and modulus of elasticity $\lambda \text{ newtons}$. The other end of the spring is attached to a fixed point A on a ceiling. The particle is hanging freely in equilibrium at a distance 1.5 m vertically below A .

(a) Find the value of λ .

(3)

The particle is now raised to the point B , where B is vertically below A and $AB = 0.8 \text{ m}$. The spring remains straight. The particle is released from rest and first comes to instantaneous rest at the point C .

(b) Find the distance AC .

(4)

1a) $P(\uparrow): T - 0.5g = 0$

$$T = 0.5g$$

using Hooke's law:

$$T = \frac{\lambda x}{L}$$

$$0.5g = \frac{\lambda (0.3)}{1.2}$$

$$\lambda = 19.6$$

1b) At B

$$E.E = \frac{\lambda x^2}{2l}$$

$$= \frac{19.6(0.4)^2}{2(1.2)}$$

$$= \frac{98}{75}$$

P.E = mgL

$$= 0.5g(0.4+x)$$

$$= 0.2g + 0.5gx$$

KE = 0

At C

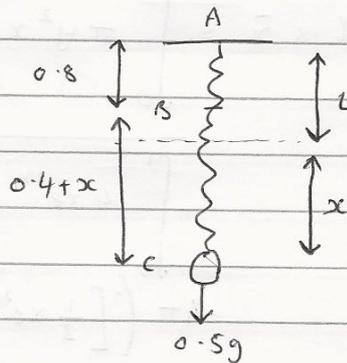
$$E.E = \frac{\lambda x^2}{2l}$$

$$= \frac{19.6 x^2}{2(1.2)}$$

$$= \frac{49}{6} x^2$$

KE = 0

PE = 0



using conservation of energy:

$$\frac{98}{75} + 0.2g + 0.5gx = \frac{49}{6} x^2$$

$$0 = \frac{49}{6} x^2 - 4.9x - \frac{49}{15}$$

$$x = 1 \quad x = -\frac{2}{5} \text{ (reject)}$$

$$\therefore AC = 1.2 + 1$$

$$= 2.2 \text{ m}$$

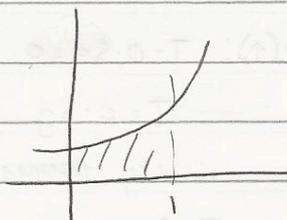
2. The finite region bounded by the x -axis, the curve with equation $y = 2e^x$, the y -axis and the line $x = 1$ is rotated through one complete revolution about the x -axis to form a uniform solid.

Use algebraic integration to

(a) show that the volume of the solid is $2\pi(e^2 - 1)$, (4)

(b) find, in terms of e , the x coordinate of the centre of mass of the solid. (6)

2a. Volume = $\int_0^1 4e^{2x} \pi dx$
 $= \pi [2e^{2x}]_0^1$



$= \pi [2e^2 - 2e^0]$
 $= 2\pi [e^2 - 1]$

2b. vol $\times \bar{x} = \int_0^1 \pi y^2 x dx$

let $u = x$ let $\frac{dv}{dx} = e^{2x}$
 $\frac{du}{dx} = 1$ $v = \frac{1}{2} e^{2x}$

$= \pi \int_0^1 4xe^{2x} dx$
 $= 4\pi \left(\left[\frac{1}{2} xe^{2x} \right]_0^1 - \int_0^1 \frac{1}{2} e^{2x} dx \right)$

$= 4\pi \left(\frac{1}{2} e^2 - \frac{1}{4} [e^{2x}]_0^1 \right)$

$= 4\pi \left(\frac{1}{2} e^2 - \frac{1}{4} (e^2 - 1) \right)$

$= 4\pi \left(\frac{1}{2} e^2 - \frac{1}{4} e^2 + \frac{1}{4} \right)$

$= \pi (e^2 + 1)$

$\bar{x} = \frac{\pi (e^2 + 1)}{2\pi (e^2 - 1)}$

$= \frac{e^2 + 1}{2(e^2 - 1)}$

$= \frac{e^2 + 1}{2(e^2 - 1)}$

3.

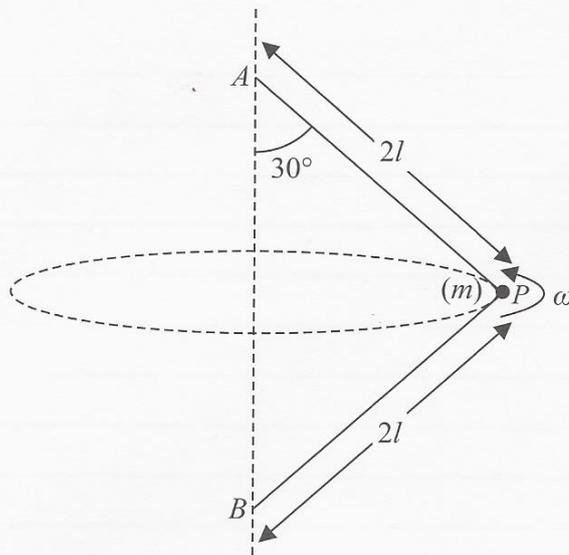


Figure 1

A small ball P of mass m is attached to the midpoint of a light inextensible string of length $4l$. The ends of the string are attached to fixed points A and B , where A is vertically above B . Both strings are taut and AP makes an angle of 30° with AB , as shown in Figure 1. The ball is moving in a horizontal circle with constant angular speed ω .

(a) Find, in terms of m , g , l and ω ,

(i) the tension in AP ,

(ii) the tension in BP .

(8)

(b) Show that $\omega^2 \geq \frac{g\sqrt{3}}{3l}$.

(2)

3: Let Tension in $AP = T_A$
 Tension in $BP = T_B$

R(\uparrow): $T_A \cos 30 = mg + T_B \cos 30$
 $T_A \left(\frac{\sqrt{3}}{2}\right) = mg + T_B \left(\frac{\sqrt{3}}{2}\right)$ (2)

R(\leftarrow): $T_A \sin 30 + T_B \sin 30 = m(r\omega^2)$
 $\frac{1}{2} T_A + \frac{1}{2} T_B = ml\omega^2$
 $T_A = 2ml\omega^2 - T_B$ (1)

Substitute (1) into (2)
 $(2ml\omega^2 - T_B) \left(\frac{\sqrt{3}}{2}\right) = mg + T_B \frac{\sqrt{3}}{2}$
 $ml\omega^2 \sqrt{3} = mg + T_B \sqrt{3}$
 $T_B = ml\omega^2 - \frac{mg}{\sqrt{3}}$

$$\therefore T_A = 2ml\omega^2 - \left(ml\omega^2 - \frac{mg}{\sqrt{3}} \right)$$

$$= ml\omega^2 + \frac{mg}{\sqrt{3}}$$

6. $T_B \geq 0$

$$ml\omega^2 - \frac{mg}{\sqrt{3}} \geq 0$$

$$l\omega^2 \geq \frac{g}{\sqrt{3}}$$

$$\omega^2 \geq \frac{g}{l\sqrt{3}}$$

$$\omega \geq \frac{\sqrt{3g}}{3l}$$

4. A vehicle of mass 900 kg moves along a straight horizontal road. At time t seconds the resultant force acting on the vehicle has magnitude $\frac{63000}{kt^2}$ N, where k is a positive constant. The force acts in the direction of motion of the vehicle. At time t seconds, $t \geq 1$, the speed of the vehicle is v m s⁻¹ and the vehicle is a distance x metres from a fixed point O on the road. When $t = 1$ the vehicle is at rest at O and when $t = 4$ the speed of the vehicle is 10.5 m s⁻¹.

(a) Show that $v = 14 - \frac{14}{t}$ (7)

(b) Hence deduce that the speed of the vehicle never reaches 14 m s⁻¹. (1)

(c) Use the trapezium rule, with 4 equal intervals, to estimate the value of x when $v = 7$ (4)

4. $R(\rightarrow)$: $\frac{63000}{kt^2} = 900 a.$

$\frac{63000}{kt^2} = 900 \frac{dv}{dt}$

$\int \frac{63000}{kt^2} dt = \int 900 dv.$

$\frac{-63000}{kt} = 900v + c$

when $t=1, v=0.$

$\frac{-63000}{k} = c.$

$\frac{-63000}{kt} = 900v - \frac{63000}{k}$

$$\text{when } t = 4, v = 10.5.$$

$$\frac{-15750}{R} = 900(10.5) - \frac{63000}{R}$$

$$\frac{47250}{R} = 9450$$

$$R = 5.$$

$$\frac{-63000}{5t} = 900v - \frac{63000}{5}$$

$$\frac{-12600}{t} = 900v - 12600$$

$$14 - \frac{14}{t} = v.$$

4b. As t is positive, for large values of T , the velocity is just below 14 ms^{-1} . It is an asymptote.

$$4c. \text{ when } v = 7$$

$$14 - \frac{14}{t} = 7.$$

$$\therefore 7 = \frac{14}{t}.$$

$$t = 2.$$

$$\therefore \frac{2-1}{4} = 0.25.$$

t	1	1.25	1.50	1.75	2.00
v	0	2.8	$\frac{14}{3}$	6	7

$$\therefore x \approx \frac{1}{2}(0.25)\left(0+7+2\left(2.8+\frac{14}{3}+6\right)\right) \approx 4.24 \text{ m (3 sf)}$$

5.

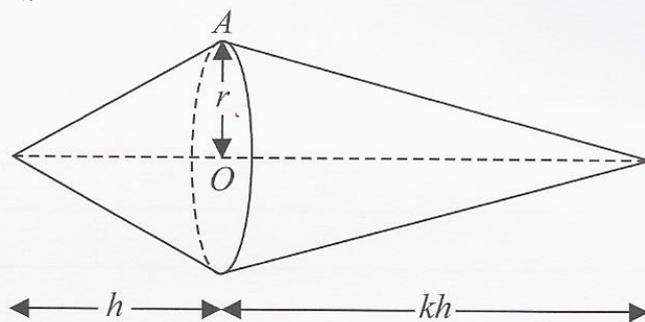


Figure 2

Figure 2 shows a uniform solid spindle which is made by joining together the circular faces of two right circular cones. The common circular face has radius r and centre O . The smaller cone has height h and the larger cone has height kh . The point A lies on the circumference of the common circular face. The spindle is suspended from A and hangs freely in equilibrium with AO at an angle of 30° to the vertical.

Show that $k = \frac{4r}{h\sqrt{3}} + 1$

(6)

5. Cone 1

$$\text{mass} = \frac{1}{3} \pi r^2 h \rho$$

$$= \frac{1}{3} \pi r^2 k h \rho$$

$$\text{Centre} = \frac{1}{4} h$$

$$= \frac{r h}{4}$$

$$= \left(\frac{r h}{4}, 0 \right)$$

Mass ratio = k

Cone 2

$$\text{mass} = \frac{1}{3} \pi r^2 h \rho$$

$$\text{Centre} = \frac{1}{4} h$$

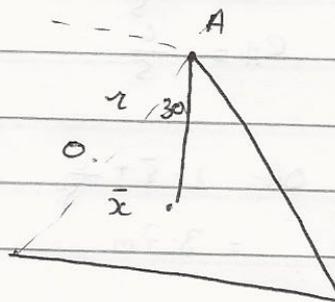
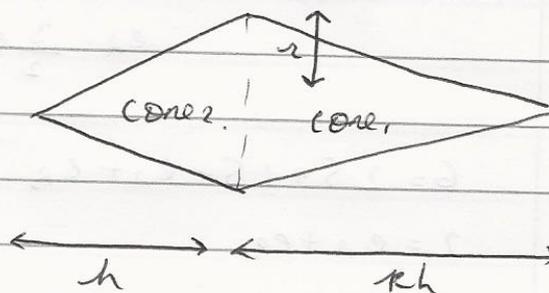
$$= \left(-\frac{1}{4} h, 0 \right)$$

mass ratio = 1.

$$k \left(\frac{r h}{4} \right) + \left(-\frac{1}{4} h \right) (1) = (k+1) (\bar{x})$$

$$\frac{k^2 h}{4} - \frac{h}{4} = (k+1) \bar{x}$$

$$\bar{x} = \frac{h(k^2 - 1)}{4(k+1)}$$



$$\therefore \tan 30 = \frac{h(k^2 - 1)}{4k(k+1)}$$

$$\frac{1}{\sqrt{3}} = \frac{h(k^2 - 1)}{4k}$$

$$\frac{4k}{h\sqrt{3}} + 1 = k$$

6.

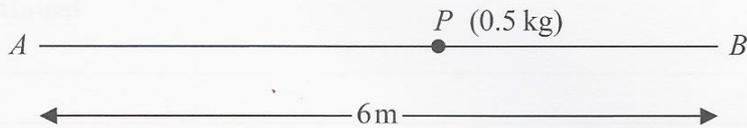


Figure 3

Two points A and B are 6 m apart on a smooth horizontal floor. A particle P of mass 0.5 kg is attached to one end of a light elastic spring, of natural length 2.5 m and modulus of elasticity 20 N. The other end of the spring is attached to A . A second light elastic spring, of natural length 1.5 m and modulus of elasticity 18 N, has one end attached to P and the other end attached to B , as shown in Figure 3. Initially P rests in equilibrium at the point O , where AOB is a straight line.

(a) Find the length of AO . (4)

The particle P now receives an impulse of magnitude 6 N s acting in the direction OB and P starts to move towards B .

(b) Show that P moves with simple harmonic motion about O . (4)

(c) Find the amplitude of the motion. (4)

(d) Find the time taken by P to travel 1.2 m from O . (3)

6.

6m.

A ————— P (0.5 kg) ————— B

←———— 2.5 ————— e_A ————— e_B ————— 1.5 —————→

$\lambda = 20$ $\lambda = 18$

Let extension of $AO = e_A$, extension of $OB = e_B$.

<p><u>For OA</u></p> $T = \frac{\lambda x}{l}$ $= \frac{20}{2.5} e_A$ $= 8e_A$	<p><u>For OB</u></p> $T = \frac{\lambda x}{l}$ $= \frac{18}{1.5} e_B$ $= 12e_B$
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R(=): $8e_A = 12e_B$

$\therefore e_A = \frac{3}{2} e_B$

$$6 = 2.5 + 1.5 + e_A + e_B$$

$$2 = e_A + e_B$$

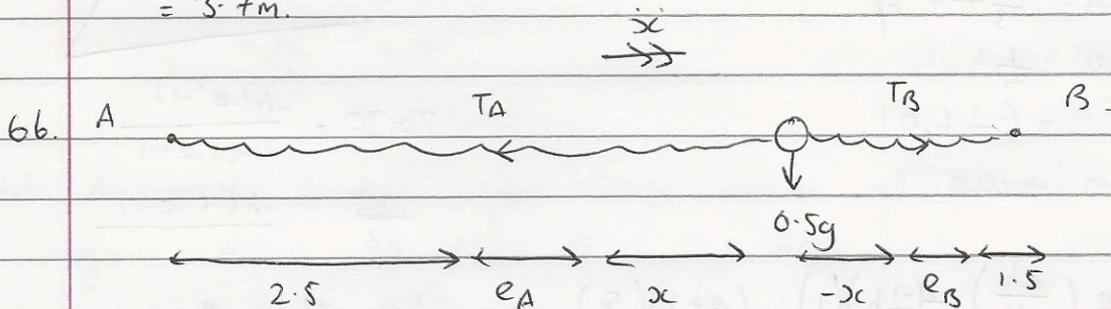
$$2 = \frac{3}{2}e_B + e_B$$

$$e_B = \frac{4}{5}$$

$$e_A = \frac{6}{5}$$

$$OA = 2.5 + \frac{6}{5}$$

$$= 3.7 \text{ m}$$



For OA

Using Hooke's law: $T_A = \frac{\lambda x}{l}$

$$= \frac{20(e_A + x)}{2.5}$$

$$= 8(e_A + x)$$

For OB

Using Hooke's law: $T_B = \frac{\lambda x}{l}$

$$= \frac{18(e_B - x)}{1.5}$$

$$= 12(e_B - x)$$

$P(\rightarrow)$: $12(e_B - x) - 8(e_A + x) = 0.5 \ddot{x}$

$$\frac{48}{5} - 12x - \frac{48}{5} - 8x = 0.5 \ddot{x}$$

$$-40x = \ddot{x}$$

\therefore SHM.

$$c. \quad I = m(v-u)$$

$$6 = 0.5(v-0)$$

$$v = 12 \text{ m s}^{-1} \rightarrow \text{max speed.}$$

$$\omega^2 = 40$$

$$\omega = 2\sqrt{10}$$

$$v = a\omega$$

$$\frac{12}{2\sqrt{10}} = a$$

$$a = 1.897366$$

$$= 1.90 \text{ (3 sf)}$$

$$d. \quad x = a \sin \omega t$$

$$1.2 = 1.90 \sin(\sqrt{40}t)$$

$$\sqrt{40}t = 0.68$$

$$t = 0.108 \text{ seconds (3 sf)}$$

7. A solid smooth sphere, with centre O and radius r , is fixed to a point A on a horizontal floor. A particle P is placed on the surface of the sphere at the point B , where B is vertically above A . The particle is projected horizontally from B with speed $\frac{\sqrt{gr}}{2}$ and starts to move on the surface of the sphere. When OP makes an angle θ with the upward vertical and P remains in contact with the sphere, the speed of P is v .

(a) Show that $v^2 = \frac{gr}{4}(9 - 8\cos\theta)$. (4)

The particle leaves the surface of the sphere when $\theta = \alpha$.

(b) Find the value of $\cos\alpha$. (4)

After leaving the surface of the sphere, P moves freely under gravity and hits the floor at the point C .

Given that $r = 0.5$ m,

(c) find, to 2 significant figures, the distance AC . (7)

Take A to have OPE . Let mass = m .

7. At B

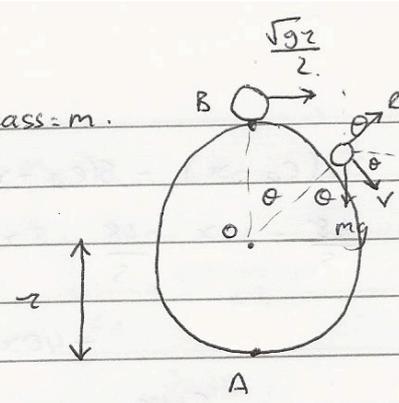
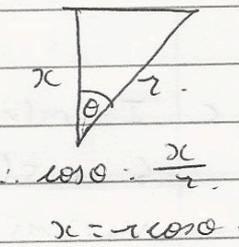
$$PE = mgh$$

$$= mg(2r)$$

$$= 2mgr$$

$$KE = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\frac{1}{4}gr\right)$$

$$= \frac{1}{8}mgr$$



At angle θ

$$PE = mgh$$

$$= mgr(1 + \cos\theta)$$

$$KE = \frac{1}{2}mv^2$$

conservation of energy: $2mgr + \frac{1}{8}mgr = mgr + mgr\cos\theta + \frac{1}{2}mv^2$

$$\frac{9}{8}gr - gr\cos\theta = \frac{1}{2}v^2$$

$$\frac{gr}{4}(9 - 8\cos\theta) = v^2$$

76. $R(\alpha)$: towards centre of sphere

$$mg \cos \theta - R = m \left(\frac{v^2}{r} \right)$$

$$R = 0.$$

$$mg \cos \theta = m \left(\frac{g^2}{4r} (9 - 8 \cos \theta) \right)$$

$$mg \cos \theta = \frac{mg}{4} (9 - 8 \cos \theta)$$

$$4 \cos \theta = 9 - 8 \cos \theta.$$

$$12 \cos \theta = 9$$

$$\cos \theta = \frac{3}{4}.$$

$$\cos \alpha = \frac{3}{4}.$$

7c. At angle α $v^2 = \frac{g^2}{4} (9 - 8 \left(\frac{3}{4} \right))$

$$v = \sqrt{\frac{3g^2}{4}}$$

Vertical component: $v \sin \alpha = \sin \alpha \sqrt{\frac{3g}{8}}$

Height above ground = $r + r \left(\frac{3}{4} \right)$
 $= \frac{7}{8}$ metres.

$$s = ut + \frac{1}{2} at^2.$$

$$\frac{7}{8} = \left(\sin \alpha \sqrt{\frac{3g}{8}} \right) t + \frac{1}{2} g (t^2)$$

$$4.9t^2 + \left(\sin \alpha \sqrt{\frac{3g}{8}} \right) t - \frac{7}{8} = 0.$$

$$t = 0.3191710 \quad 0.3125544...$$

Horizontal component: $v \cos \alpha = \cos \alpha \sqrt{\frac{3g}{8}}$

distance travelled in time $t = 0.3125544...$

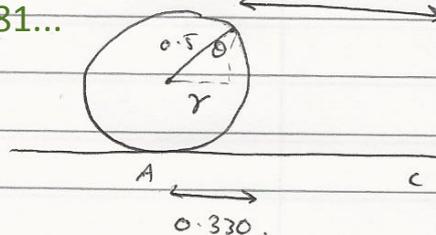
$$v = \frac{s}{t}.$$

$$s = \cos \alpha \sqrt{\frac{3g}{8}} \times 0.3125544... \quad 0.449381...$$

$$= 0.4588... \quad 0.449381...$$

$$0.449381...$$

~~$$0.4588$$~~



$$\therefore \sin \theta = \frac{y}{0.5}$$

$$\therefore y = 0.330$$

$$0.449381 + 0.330$$

$$\therefore \text{From A to BC} = 0.4588 + 0.330...$$

~~$$0.78961$$~~

~~$$0.79 \text{ m (2 sf)}$$~~
$$0.779 \text{ (3 sf)}$$