M3 January 2017 IAL (MA)

Q1) $V = \pi \int (y^2)\, dx = 9\pi \int_0^4 (4-x)\, dx = 9\pi \left[4x - \frac{x^2}{2}\right]_0^4$

$y=0$: $x=4$

$= 9\pi[16-8] = 8(9\pi) = 72\pi$

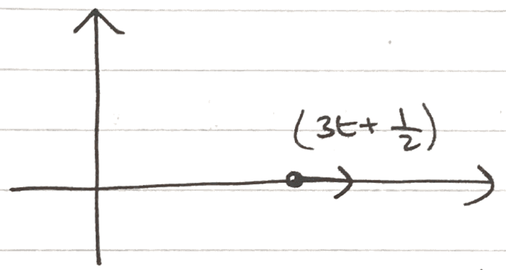
$M\bar{x} = \pi\rho \int_0^4 y^2 x\, dx = 9\pi\rho \int_0^4 (4x - x^2)\, dx$

$= 9\pi\rho\left[2x^2 - \frac{x^3}{3}\right]_0^4 = 9\pi\rho\left[32 - \frac{64}{3}\right]$

$= 96\pi\rho$

$\bar{x} = \frac{M\bar{x}}{M} = \frac{96\pi\rho}{72\pi\rho} = \boxed{\frac{4}{3}}$

Q2a)



$F = 3t + \frac{1}{2} = ma$

$3t + \frac{1}{2} = 0.6 \frac{dv}{dt}$

$\int (0.6)\, dv = \int \left(3t + \frac{1}{2}\right) dt$

$0.6v = \frac{3t^2}{2} + \frac{1}{2}t + c$

$\underline{t=0, v=0}$: $c = 0$ //

so $0.6v = \frac{3}{2}t^2 + \frac{1}{2}t$

$\underline{\div 0.6}$: $\boxed{v = \frac{5}{2}t^2 + \frac{5}{6}t}$

b) $v = \frac{10}{3}$: $\frac{10}{3} = \frac{5}{2}t^2 + \frac{5}{6}t$

$\underline{\times 6}$: $20 = 15t^2 + 5t$

$15t^2 + 5t - 20 = 0$

$3t^2 + t - 4 = 0$

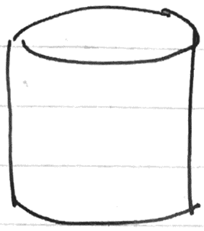
$(3t + 4)(t - 1) = 0$

$t = 1$ // $(t > 0)$

$OA = \int_0^1 (v)\, dt = \int_0^1 \left[\frac{5t^2}{2} + \frac{5t}{6}\right] dt$

$= \left[\frac{5t^3}{6} + \frac{5t^2}{12}\right]_0^1 = \frac{5}{6} + \frac{5}{12} - 0 = \boxed{\frac{5}{4}\text{m}}$

Q3)

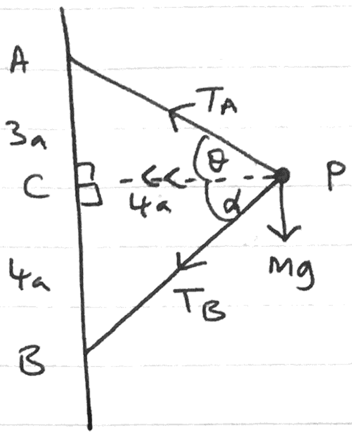
Shape	Mass (vol. ratio)	Distance of c.o.m from O
	$\pi(4a)^2(6a) = \boxed{96\pi a^3}$	$3a$
(−) 	$\frac{2}{3}\pi(3a)^3 = \boxed{18\pi a^3}$	$\frac{3(3a)}{8} = \frac{9a}{8}$
	$\boxed{78\pi a^3}$	$\bar{y}$

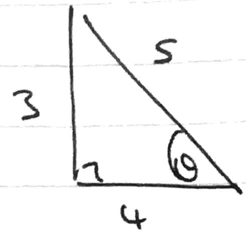
moments about diameter through O

$$96\cancel{\pi}\cancel{a^3}(3a) - 18\cancel{\pi}\cancel{a^3}\left(\frac{9a}{8}\right) = 78\cancel{\pi}\cancel{a^3}(\bar{y})$$

$$\frac{\left[96(3) - \frac{18(9)}{8}\right]a}{78} = \bar{y} = \frac{\frac{1071}{4}a}{78} = \boxed{\frac{357a}{104}}$$

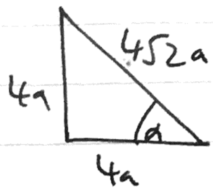
Q4a)





$\cos\theta = \frac{4}{5}$

$\sin\theta = \frac{3}{5}$



$\cos\alpha = \frac{\sqrt{2}}{2} = \sin\alpha$

$R(\updownarrow)$: $T_A \sin\theta = mg + T_B \sin\alpha$ — ①

$\overset{+}{\leftarrow}$ N2L(P) : $T_A \cos\theta + T_B \cos\alpha = m(4a)\omega^2$

$\sin\alpha = \cos\alpha$

so ① : $T_B \cos\alpha = T_A \sin\theta - mg$

and ② : $T_B \cos\alpha = 4am\omega^2 - T_A \cos\theta$

equating : $T_A\left(\frac{3}{5}\right) - mg = 4ma\omega^2 - T_A\left(\frac{4}{5}\right)$

$$T_A\left(\frac{3}{5} + \frac{4}{5}\right) = m(4a\omega^2 + g)$$

$$\frac{7}{5}T_A = m(4a\omega^2 + g)$$

$$\therefore T_A = \frac{5m}{7}(4a\omega^2 + g)$$

b) from ① : $T_B\left(\frac{\sqrt{2}}{2}\right) = T_A\left(\frac{3}{5}\right) - mg$

$$T_B \cdot \frac{\sqrt{2}}{2} = \frac{3}{5}\left(\frac{5m}{7}\right)(4a\omega^2 + g) - mg$$

$$T_B \cdot \frac{\sqrt{2}}{2} = \frac{3m}{7}(4a\omega^2 + g) - mg$$

$$T_B \cdot \frac{\sqrt{2}}{2} = \frac{12ma\omega^2}{7} + \frac{3mg}{7} - mg$$

$$T_B \cdot \frac{\sqrt{2}}{2} = \frac{12ma\omega^2}{7} - \frac{4mg}{7}$$

$$T_B \cdot \frac{\sqrt{2}}{2} = \frac{4m}{7}(3a\omega^2 - g)$$

$$\therefore T_B = \frac{4m \times 2}{7\sqrt{2}}(3a\omega^2 - g)$$

$$= \boxed{\frac{4m\sqrt{2}}{7}(3a\omega^2 - g)}$$

c) $T_B \geq 0$ for P to continue its circular motion :

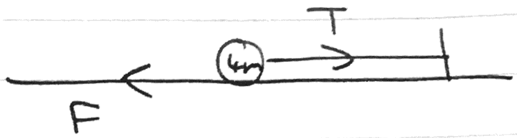
$$\Rightarrow \frac{4m\sqrt{2}}{7}(3a\omega^2 - g) \geq 0$$

$$\Rightarrow 3a\omega^2 - g \geq 0$$

$$\Rightarrow \omega^2 \geq \frac{g}{3a} \quad \therefore \boxed{\omega \geq \sqrt{\frac{g}{3a}}}$$

$(k = 3)$ //

Q5a)



$$F \leq \mu R \longrightarrow \mu \geq \frac{F}{R}$$

since P is at rest, $F = T$.

$$T = \frac{\lambda x}{l} = \frac{3mg\left(\frac{l}{3}\right)}{l} = mg$$

so $F = mg$. and $R = 4mg$.
(resolving perp to ground)

and $\mu \geq \frac{F}{R} \rightarrow \mu \geq \frac{\cancel{m}g}{4\cancel{m}\cancel{g}}$

hence $\mu \geq \frac{1}{4}$

b) Initially: $KE = 0$
$EPE = \frac{3mg}{2l}(l)^2$

When P comes to rest: $KE = 0$
$EPE = \frac{3mg}{2l}(l-d)^2$

W.D by friction $= \frac{2}{5}(4mg)d$

C.O.E : $\frac{3mgl}{2} = \frac{3mg}{2l}(l^2 + d^2 - 2dl) + \frac{8}{5}mgd$

$$\frac{3mgl}{2} = \frac{3mgl}{2} + \frac{3mgd^2}{2l} - 3mgd + \frac{8}{5}mgd$$

$$\frac{3mgd^2}{2l} = \frac{7}{5}mgd$$

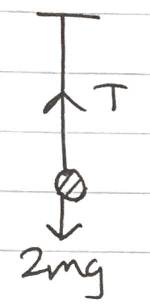
$$mgd\left(\frac{3d}{2l} - \frac{7}{5}\right) = 0$$

$$d \neq 0, \quad \frac{3d}{2l} - \frac{7}{5} = 0$$

$$\underline{\underline{d = \frac{14}{15}l}} \quad < 0$$ hence P moves with S.H.M cen'

hence P comes to rest before the string becomes slack.

Q6a)

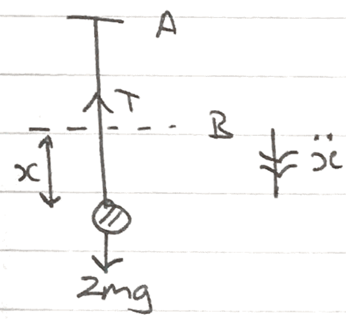


$$T = 2mg. \qquad T = \frac{\lambda x}{l} = \frac{20mge}{5l}$$

$$\frac{20mge}{5l} = 2mg$$

$$\therefore e = \frac{10l}{20} = \boxed{\frac{l}{2}}$$

b) $\underline{N2L(P)}\downarrow_+$: $2mg - T = 2m\ddot{x}$



$$T = \frac{20mg}{5l}\left(x + \frac{l}{2}\right)$$

$$2mg - \frac{20mgx}{5l} - 2mg = 2m\ddot{x}$$

$$-\frac{4mgx}{l} = 2m\ddot{x}$$

$\underline{\div 2m}$: $-\frac{2g}{l}x = \ddot{x}$ hence P moves with S.H.M centre B.

c) $\omega = \sqrt{\frac{2g}{l}}$ $\therefore T = \boxed{2\pi\sqrt{\frac{l}{2g}}}$

$\left(T = \frac{2\pi}{\omega}\right)$

d) $V_{max} = a\omega = \frac{\sqrt{gl}}{5} = a\sqrt{\frac{2g}{l}}$

$\therefore a = \frac{\cancel{\sqrt{g}}(l)}{(5\sqrt{2})\cancel{\sqrt{g}}} = \boxed{\frac{l}{5\sqrt{2}}}$

e) $x = \frac{l}{5\sqrt{2}}\cos\left(\sqrt{\frac{2g}{l}}t\right) = a\cos\left(t\sqrt{\frac{2g}{l}}\right)$

$\frac{a}{2} = a\cos\left(t\sqrt{\frac{2g}{l}}\right)$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = t\sqrt{\frac{2g}{l}} = \frac{\pi}{3}$

$\therefore t = \frac{\pi}{3}\sqrt{\frac{l}{2g}}$:= time to D.

P won't be slack at any point in the motion as $\frac{l}{5\sqrt{2}} < \frac{l}{2}$.

$\underline{\text{time to E}}$: $-\cancel{a} = \cancel{a}\cos\left(t\sqrt{\frac{2g}{l}}\right)$

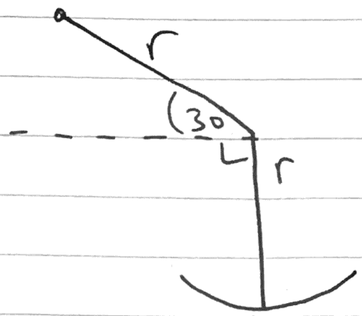
$\cos^{-1}(-1) = t\sqrt{\frac{2g}{l}} = \pi$

$\therefore t = \pi\sqrt{\frac{l}{2g}}$

hence time required $= \left(\pi - \frac{\pi}{3}\right)\sqrt{\frac{l}{2g}}$

(= time to E − time to D) $= \frac{2\pi}{3}\sqrt{\frac{l}{2g}}$

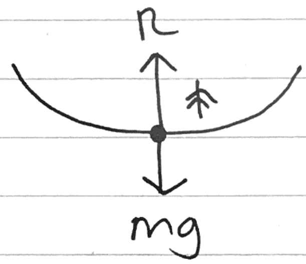
Q7a) Energy to B : $\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}_{\Delta KE} = \underbrace{mgr(1+\sin 30)}_{\Delta GPE}$



$$\frac{v^2}{2} = \frac{u^2}{2} + \frac{3gr}{2}$$

$$v^2 = u^2 + 3gr$$

At B :

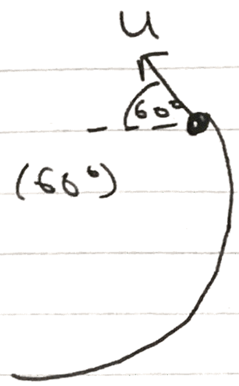


N2L(P) : $R - mg = \frac{m}{r}(v^2)$

$$R - mg = \frac{mu^2}{r} + \frac{3mgr}{r}$$

$$R = mg + \frac{mu^2}{r} + 3mg$$

$$\boxed{R = 4mg + \frac{mu^2}{r}}$$



b) P will leave the bowl at the same angle to the horizontal (60°) it started at.

$$\left.\begin{aligned} s &= d \\ u &= u\sin 60 \\ v &= 0 \\ a &= -g \\ t &= \end{aligned}\right\} \quad \begin{aligned} v^2 &= u^2 + 2as \\ 0 &= u^2\sin^2 60 - 2gd \end{aligned}$$

$$d = \frac{u^2\sin^2 60}{2g} = \boxed{\frac{3u^2}{8g}}$$

$\therefore$ greatest height above floor $= d + \frac{3r}{2}$

$$= \boxed{\frac{3u^2}{8g} + \frac{3r}{2}}$$

c) finding time when P returns

$$\left.\begin{aligned} s &= 0 \\ u &= u\sin 60 \\ v &= -u\sin 60 \\ a &= -g \\ t &= t \end{aligned}\right\} \quad \begin{aligned} v &= u + at \\ -u\sin 60 &= u\sin 60 - gt \\ t &= \frac{2u\sin 60}{g} \end{aligned}$$

horizontal distance travelled (in this time) $= u\cos 60 \times \frac{2u\sin 60}{g}$

$$= \frac{2u^2\sqrt{3}}{4g} = \frac{u^2\sqrt{3}}{2g}$$

[bowl width = $2r\cos 30$]

but $u^2 > 2gr$: $\overleftarrow{\text{distance}} > \frac{2gr}{2g}\sqrt{3}$

$\Rightarrow \overleftarrow{\text{distance}} > r\sqrt{3}$

hence $\overleftarrow{\text{distance}}$ > bowl width

∴ P doesn't fall back in.