M3 June 2017 (MA)

Q1) Area = $\int_0^2 (4 - x^2)\,dx$ (at $y=0, x=2$)

$= \left[4x - \frac{x^3}{3}\right]_0^2 = \left[8 - \frac{8}{3}\right] - [0] = \boxed{\frac{16}{3}}$

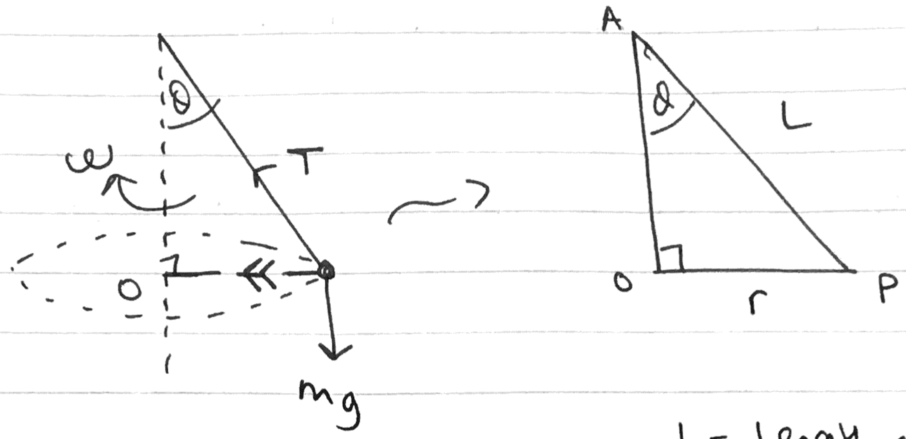
so mass $= M = \frac{16\rho}{3}$ (where $\frac{\text{mass}}{\text{area}} = \rho$)

$M\bar{x} = \rho\int_0^2 (xy)\,dx = \rho\int_0^2 \left[4x - x^3\right] dx = \rho\left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= \rho\left[2(4) - \frac{2^4}{4}\right] - [0] = \boxed{4\rho}$

so $\bar{x} = \frac{M\bar{x}}{M} = \frac{4\cancel{\rho}}{\frac{16}{3}\cancel{\rho}} = \boxed{\frac{3}{4}}$

Q2 i) / ii



l = length of string.

$R(\updownarrow)$: $T\cos\theta = mg$

$1.2mg\cos\theta = mg$

$\therefore \cos\theta = \frac{1}{1.2} = \frac{5}{6}$.

$(\overset{+}{\leftrightarrow})$ N2L(P) : $T\sin\theta = mr\omega^2$

$1.2\cancel{m}g\sin\theta = \cancel{m}r(58.8)$

$\cos\theta = \frac{5}{6}$ so $\sin\theta = \sqrt{1-\left(\frac{5}{6}\right)^2} = \frac{\sqrt{11}}{6}$.

$\Rightarrow \frac{1.2g\sqrt{11}}{6} = 58.8r$

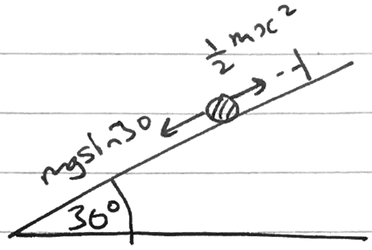
$\Rightarrow r = \frac{1.2g\sqrt{11}}{6 \times 58.8} = \underline{\underline{0.111\text{m}}}$

so $L\sin\theta = 0.111\ldots$

$L = \frac{0.111\ldots}{\frac{\sqrt{11}}{6}} = \boxed{0.2\text{m}}$

and $\theta = \cos^{-1}\frac{5}{6} = \boxed{33.6°}$

Q3a)



$\swarrow +$ N2L (P) : $\frac{mg}{2} - \frac{1}{2}mx^2 = ma$

$$\frac{mg}{2} - \frac{mx^2}{2} = mv\frac{dv}{dx}$$

$$g - x^2 = 2v\frac{dv}{dx}$$

$$\int(g - x^2)\,dx = 2\int(v)\,dv$$

$$v^2 = gx - \frac{x^3}{3} + c$$

$x = 0, v = 0$: $0 = 0 + c$

$\therefore c = 0$

$$\text{so } v^2 = gx - \frac{x^3}{3}$$

$x = 3$: $v^2 = 3g - \frac{27}{3} = \frac{102}{5}$

$$v = \sqrt{\frac{102}{5}} = \boxed{4.52\,m/s}$$

b) $v^2 = gx - \frac{x^3}{3}$

$\underline{v=0}$: $gx - \frac{x^3}{3} = 0$

$$x\left(g - \frac{x^2}{3}\right) = 0$$

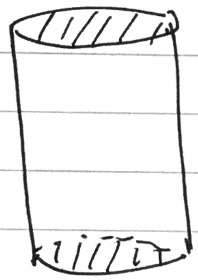
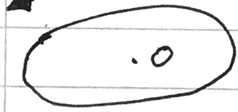
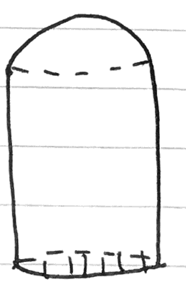
$x \neq 0$
[when P comes to rest again]

$$\therefore g = \frac{x^2}{3}$$

$$x^2 = 3g$$

$$\therefore x = \sqrt{3g} \simeq \boxed{5.4m}$$

Q4a)

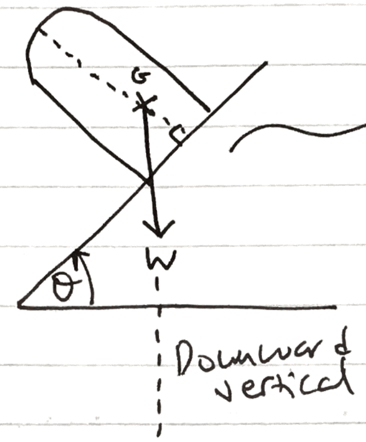
	Shape	Mass (surface area)	Distance of c.o.m from O
+		$\boxed{2\pi a^2}$	$\boxed{\frac{9a}{2}}$
+		$2\pi r h = \boxed{8\pi a^2}$	$\boxed{2a}$
base +		$\boxed{\pi a^2}$	$\boxed{0}$
=		$(8+2+1)\pi a^2 = \boxed{11\pi a^2}$	$\boxed{\bar{y}}$

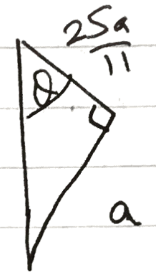
taking moments about diameter of base...

$$2\pi a^2\left(\frac{9a}{2}\right) + 8\pi a^2(2a) + \pi a^2(0) = 11\pi a^2(\bar{y})$$

$$\frac{9a + 16a}{11} = \bar{y} = \boxed{\frac{25a}{11}}$$

b)





$$\tan\theta = \frac{a}{\frac{25a}{11}} = \frac{11}{25}$$

$$\therefore \theta = \tan^{-1}\frac{11}{25} = \boxed{23.7°}$$

[on the point of toppling ∴ downward vertical passes through lowest point of contact with the plane]

(Q5a) <u>at A</u> : $KE = \frac{1}{2}m(7ag)$

$GPE = mga$

<u>at B angle θ to horizontal</u> : $KE = \frac{1}{2}mv^2$

$GPE = mga(1-\sin\theta)$

(Lowest point = 0 GPE level)

<u>C.O.E</u> : $\frac{7amg}{2} + a\cancel{mg} = \frac{mv^2}{2} + a\cancel{mg} - amg\sin\theta$

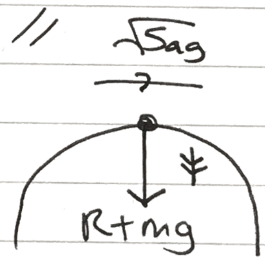
$mv^2 = 7amg + 2amg\sin\theta$

<u>$\div m$</u> : $v^2 = 7ag + 2ag\sin\theta$

$v^2 = ag(7 + 2\sin\theta)$

b) at top, $\theta = 270°$: $v^2 = ag(7 + 2\sin 270)$

$v^2 = 5ag$ //



$+\downarrow$ <u>N2L(P at top)</u> : $R + mg = \frac{mv^2}{a}$

$\Rightarrow R + mg = \frac{m}{a}(5ag)$

$\Rightarrow R + mg = 5mg$

$\Rightarrow R = 4mg$ // so $R > 0$ at top hence P remains in contact with the surface.

so P will move in a complete circle.

c) V will be max at the lowest point since this is when GPE is minimum.

so at $\theta = 90°$: $V^2 = 9ag$

$\therefore V_{max} = \boxed{3\sqrt{ag}}$

alt: $V^2 = ag(7 + 2\sin\theta)$

so V will be max when $\sin\theta = 1$.

hence $V_{max}^2 = ag(7+2)$

$= 9ag$

(so $V = 3\sqrt{ag}$)

Q6a) $T = \frac{\lambda x}{l}$

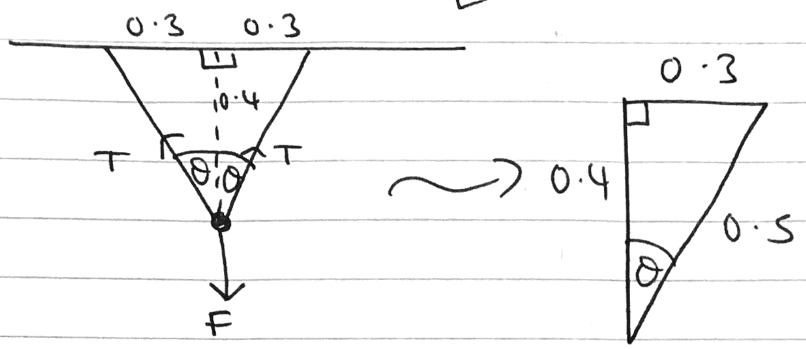
$8 = \frac{\lambda(0.2)}{0.4}$

$8 = \frac{\lambda}{2}$

×2: $\lambda = 16$ (N)

[no weight acts parallel to this horizontal plane]

b)



$\cos\theta = \frac{4}{5}$

$+\uparrow$ N2L (P) : $2T\cos\theta - F = 0$

$2T\cos\theta = F$

$T = \frac{\lambda x}{l} = \frac{16}{0.2}(0.5 - 0.2) = 24N$ //

for one half of the string

so $2(24)\left(\frac{4}{5}\right) = F = \boxed{38.4N}$

c) using energy : initially : $KE = 0$

$EPE = \frac{16}{0.8}(1 - 0.4)^2$

at AB : $KE = 0.15v^2$

$EPE = \frac{\lambda x^2}{2l} = 8 \times \left(\frac{0.2}{2}\right)$

$= 0.8N$

C.O.E : $7.2 = 0.15v^2 + 0.8$

$v^2 = \frac{7.2 - 0.8}{0.15} = \frac{128}{3}$

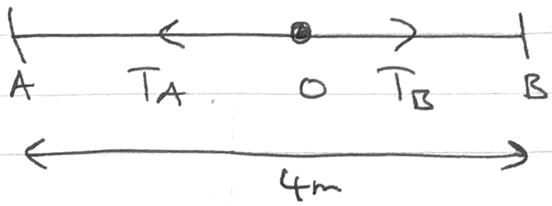
[remember there is no change in GPE as P is moving only in the horizontal plane]

$v = \sqrt{\frac{128}{3}} = \boxed{6.53 m/s}$

Q7a)

$l = 1.8$ $\qquad$ $l = 1.2$

$\lambda = 45$ $\qquad$ $\lambda = 20$



$$T_A = T_B$$

$$\frac{45}{1.8}(AO - 1.8) = \frac{20}{1.2}(4 - AO - 1.2)$$

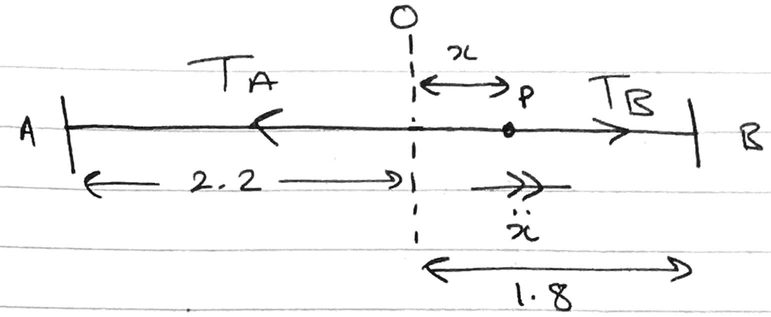
$$25AO - 45 = \frac{50}{3}(2.8 - AO)$$

$$25AO - 45 = \frac{140}{3} - \frac{50}{3}AO$$

$$AO\left(25 + \frac{50}{3}\right) = \frac{140}{3} + 45$$

$$AO = \frac{\frac{140}{3} + 45}{\frac{50}{3} + 25} = \boxed{2.2m}$$

b)



$\xrightarrow{+}$

N2L(P): $T_B - T_A = 0.6\ddot{x}$

$T_B = \frac{20}{1.2}(1.8 - x - 1.2)$

$T_A = \frac{45}{1.8}(2.2 + x - 1.8)$

$T_B = 10 - \frac{50}{3}x$

$T_A = 10 + 25x$

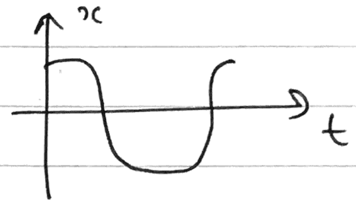
$\therefore T_B - T_A = 10 - 10 - \frac{125}{3}x$

$\therefore -\frac{125}{3}x = 0.6\ddot{x}$

$\Rightarrow \ddot{x} = -\frac{625}{9}x$

finding where $\ddot{x}$ is positive

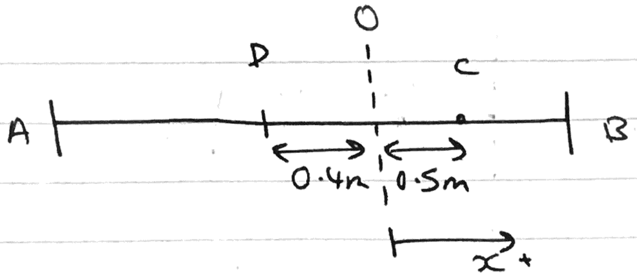
P starts at an endpoint (C). so $x = a\cos\omega t$ applies.



as seen on the graph x is max at $t = 0$. so this means that x must be increasing in the direction OB. so $\ddot{x}$ is also increasing in the direction OB. so $\ddot{x}$ is positive in this direction.

hence P moves with S.H.M centre O.

c)



$$\ddot{x} = -\frac{625}{9}x$$

$$\therefore \omega = \sqrt{\frac{625}{9}} = \frac{25}{3}.$$

and $a = 0{\cdot}5$ as $OC = 0{\cdot}5\text{m}$

$$\therefore \quad x = 0{\cdot}5\cos\left(\frac{25t}{3}\right)$$

at D, $x = -0{\cdot}4$

$$-0{\cdot}4 = 0{\cdot}5\cos\left(\frac{25t}{3}\right)$$

$$-\frac{4}{5} = \cos\left(\frac{25t}{3}\right)$$

$$\frac{25t}{3} = \cos^{-1}\left(-\frac{4}{5}\right)$$

$$\therefore t = \frac{3}{25}\cos^{-1}\left(-\frac{4}{5}\right)$$

= time from C to D.

<u>from D to A</u> : P will be influenced by T_A until AP = 1·8m where AP will become slack.

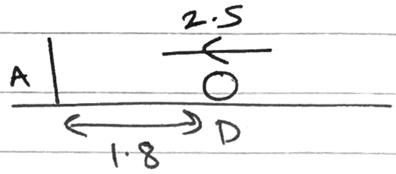
But at D, AP = 1·8m so from D to A, P will travel at constant speed.

we need to find this 'constant speed'.

recall $x = 0.5\cos\left(\frac{25t}{3}\right)$

then $\dot{x} = -\frac{25}{6}\sin\left(\frac{25t}{3}\right)$

at $t = \frac{3}{25}\cos^{-1}\left(-\frac{4}{5}\right)$: $|\dot{x}| = 2.5\ \text{ms}^{-1}$ //



$s = ut$

$1.8 = 2.5t$

$\therefore t = \frac{1.8}{2.5}$ = time from D to A.

hence total time from C to A $= \frac{3}{25}\cos^{-1}\left(-\frac{4}{5}\right) + \frac{1.8}{2.5}$

$= 1.01977\ldots$ s

$= \boxed{1.02}$ sec