

C12 Jan 2014 (MA)

$$\begin{aligned} \text{Q1)} \quad \left(2 - \frac{x}{2}\right)^6 &\approx (2)^6 + \binom{6}{1}(2^5)\left(-\frac{x}{2}\right)^1 + \binom{6}{2}(2^4)\left(-\frac{x}{2}\right)^2 \\ &\approx 64 + 192\left(-\frac{x}{2}\right) + 240\left(\frac{x^2}{4}\right) + \dots \end{aligned}$$

$$\approx 64 - 96x + 60x^2$$

$$\text{Q2a)} \quad f(x) = \frac{8}{x^2} - 4x^{\frac{1}{2}} + 3x - 1$$

$$\therefore = 8x^{-2} - 4x^{\frac{1}{2}} + 3x - 1$$

$$\therefore f'(x) = (-2)(8x^{-3}) - 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 3$$

$$= \frac{-16}{x^3} - \frac{2}{x^{\frac{1}{2}}} + 3$$

$$\text{b)} \quad \int f(x) dx = \int 8x^{-2} - 4x^{\frac{1}{2}} + 3x - 1 dx$$

$$= \frac{8x^{-1}}{-1} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^2}{2} - x + C$$

$$= -\frac{8}{x} - \frac{8}{3}x^{\frac{3}{2}} + \frac{3}{2}x^2 - x + C$$

3a) substituting  $x = 2$  into  $f(x)$ :

$$f(2) = 10(2^3) + 27(2^2) - 13(2) - 12 = \boxed{150} \quad \leftarrow \text{remainder}$$

ii) substituting  $x = -3$

$$f(-3) = 10(-3)^3 + 27(-3)^2 - 13(-3) - 12 = \boxed{0}$$

$\therefore$  there is no remainder when  $f(x)$  is divided by  $(x+3)$ .

b) from (ii), the remainder was 0 which means  $(x+3)$  is a factor of  $f(x)$ .

using long division to obtain the 'quotient':

$$\begin{array}{r} 10x^2 - 3x - 4 \\ x+3 \overline{) 10x^3 + 27x^2 - 13x - 12} \\ \underline{(-) 10x^3 + 30x^2} \end{array}$$

$$\begin{array}{r} 0 - 3x^2 - 13x \\ \underline{(-) -3x^2 - 9x} \\ 0 - 4x - 12 \\ \underline{-4x - 12} \\ 0 \quad 0 \end{array}$$

$$\therefore f(x) = (x+3)(10x^2 - 3x - 4)$$

$$\text{but } (10x^2 - 3x - 4) = (5x - 4)(2x + 1)$$

$$\text{so } f(x) = \boxed{(x+3)(5x-4)(2x+1)}$$

4i) rationalising the fraction...  $(2\sqrt{2})^2 = 8$ .

$$\frac{4}{2\sqrt{2} - \sqrt{6}} = \frac{4(2\sqrt{2} + \sqrt{6})}{(2\sqrt{2} - \sqrt{6})(2\sqrt{2} + \sqrt{6})}$$

$$= \frac{8\sqrt{2} + 4\sqrt{6}}{8 + 4\cancel{\sqrt{3}} - 4\cancel{\sqrt{3}} - 6} = \frac{8\sqrt{2} + 4\sqrt{6}}{2}$$

$$= 4\sqrt{2} + 2\sqrt{6} = \underline{2\sqrt{2}(2 + \sqrt{3})}$$

check:  $2\sqrt{2}(2 + \sqrt{3}) = 4\sqrt{2} + 2\sqrt{2} \times \sqrt{3} = 4\sqrt{2} + 2\sqrt{6} \quad \checkmark$

ii) LHS =  $(\sqrt{27}) + (\sqrt{21} \times \sqrt{7}) - \frac{6}{\sqrt{3}}$

$$= \sqrt{9 \times 3} + (\sqrt{7} \times \sqrt{7} \times \sqrt{3}) - \frac{6}{\sqrt{3}}$$

$$= 3 \times \sqrt{3} + (7 \times \sqrt{3}) - \frac{6\sqrt{3}}{3}$$

$$= 3\sqrt{3} + 7\sqrt{3} - 2\sqrt{3}$$

$$= (3 + 7 - 2)\sqrt{3} = \underline{8\sqrt{3}}$$

rationalising yields  $(-\frac{6\sqrt{3}}{3})$ .

$$5a) u_1 = 3$$

$$u_{n+1} = 2 - \frac{4}{u_n}$$

$$\therefore u_2 = 2 - \frac{4}{u_1} = 2 - \frac{4}{3} = \boxed{\frac{2}{3}}$$

$$u_3 = 2 - \frac{4}{u_2} = 2 - \frac{4}{\frac{2}{3}} = \boxed{-4}$$

$$u_4 = 2 - \frac{4}{u_3} = 2 - \frac{4}{-4} = \boxed{3}$$

b) this sequence follows a pattern...

$$\sum_1^n u_i = (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + \dots$$

as  $u_4 = 3$  and  $u_1 = 3$ , a "loop" is essentially formed.

every 3<sup>rd</sup> term after  $u_1$  is equal to 3

$$\therefore \boxed{u_{61} = 3}$$

$$c) \sum_1^{99} u_i = (3 + \frac{2}{3} - 4) + (3 + \frac{2}{3} - 4) + \dots$$

$$= 33 \times (3 + \frac{2}{3} - 4)$$

$$= 33 \times (-\frac{1}{3}) = \boxed{-11}$$

$$6) \quad ab = 25 \quad \text{---} \quad \textcircled{1}$$

$$\log_4 a - \log_4 b = 3 \quad \text{---} \quad \textcircled{2}$$

from  $\textcircled{2}$  :  $\log_4 \left( \frac{a}{b} \right) = 3 \rightarrow$  (subtraction rule)

$$\therefore 4^3 = \frac{a}{b} = 64$$

$$\therefore \underline{a = 64b} \quad \text{---} \quad \textcircled{3}$$

Substituting  $\textcircled{3}$  into  $\textcircled{1}$  :

$$(64b)(b) = 25$$

$$64b^2 = 25$$

$$b^2 = \frac{25}{64} \quad \therefore b = \boxed{\frac{5}{8}}$$

and from  $\textcircled{1}$ ,  $a = \frac{25}{\frac{5}{8}} = \boxed{40}$

7a)  $12 \sin^2 x - \cos x - 11 = 0$

$\sin^2 x = 1 - \cos^2 x$

$\therefore 12(1 - \cos^2 x) - \cos x - 11 = 0$

← (Rearrange...)

$\Rightarrow 12 \cos^2 x + \cos x - 1 = 0$

b) from (a),  $12 \cos^2 x + \cos x - 1 = 0$  //

let  $y = \cos x$ ,

$0 \leq x \leq 360^\circ$

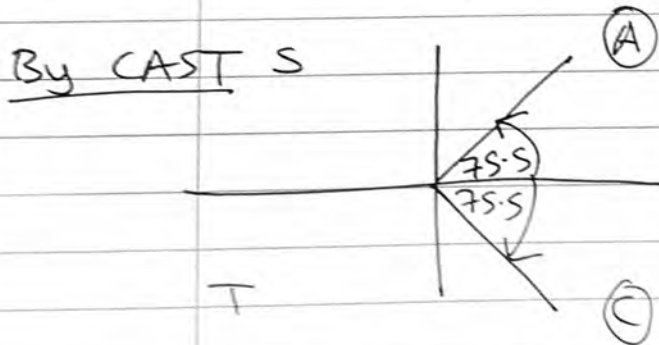
then  $12y^2 + y - 1 = 0$

$\Rightarrow (4y - 1)(3y + 1) = 0$

$\therefore y = \frac{1}{4}$

$\cos x = \frac{1}{4}$

$x = \cos^{-1}\left(\frac{1}{4}\right) = 75.5^\circ$



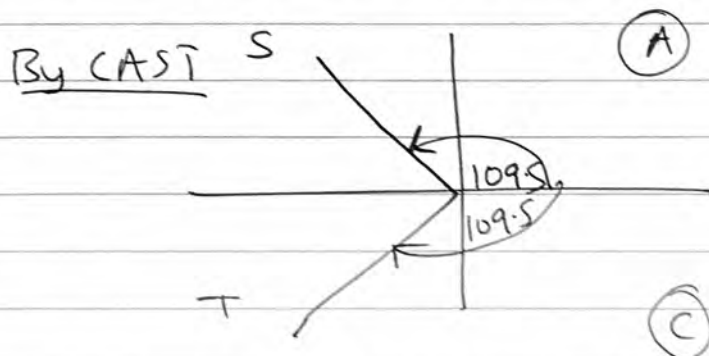
$x = 75.5^\circ, 360 - 75.5^\circ$

$\therefore x = \boxed{75.5^\circ, 284.5^\circ}$

$y = -\frac{1}{3}$

$\cos x = -\frac{1}{3}$

$x = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5^\circ$



$x = 109.5^\circ, 360 - 109.5^\circ$

$x = \boxed{109.5^\circ, 250.5^\circ}$

$$8) \quad kx^2 + 8x + 2(k+7) = 0$$

no real roots  $\therefore b^2 - 4ac < 0$

$$\left. \begin{array}{l} a = k \\ b = 8 \\ c = 2(k+7) \end{array} \right\} \begin{array}{l} b^2 - 4ac < 0 \\ (8)^2 - 4(k)(2)(k+7) < 0 \end{array}$$

$$64 - 8k(k+7) < 0$$

$$64 - 8k^2 - 56k < 0$$

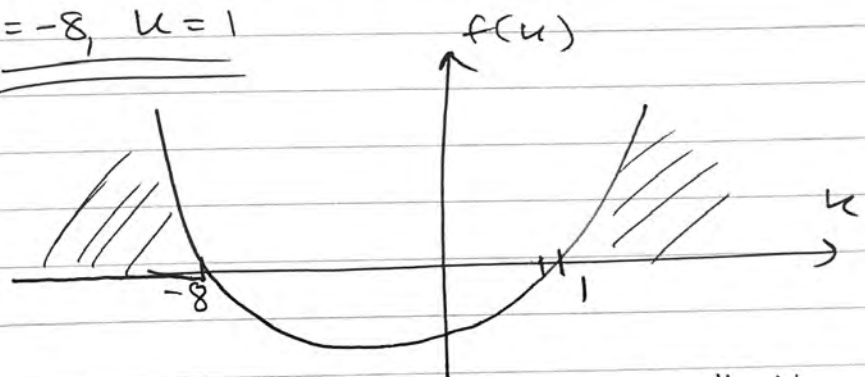
$$\Rightarrow 8k^2 + 56k - 64 > 0$$

$$\textcircled{8}: \quad k^2 + 7k - 8 > 0$$

solving  $[k^2 + 7k - 8 = 0]$  to find critical values:

$$(k+8)(k-1) = 0$$

$$\underline{\underline{k = -8, k = 1}}$$



Region required is where "y" > 0

hence values are:

$$\boxed{\begin{array}{l} k > 1 \\ k < -8 \end{array}}$$

9a)

$$a + ar + ar^2 + \dots$$

$$300 \quad 300 \times 1.05 \quad \dots$$

Number sold in 24<sup>th</sup> month = 24<sup>th</sup> term =  $ar^{23}$

$$= (300) \times (1.05)^{23} \approx 921.5$$

$$\therefore \boxed{921 \text{ phones sold}}$$

b)  $S_n = \frac{a(1-r^n)}{1-r}$

$$\therefore S_{24} = \frac{300(1 - (1.05^{24}))}{1 - 1.05} = 13350.59966\dots$$

$$\therefore \boxed{13351} \text{ to the nearest integer}$$

c)  $300(1.05)^{n-1} > 3000$

↖ number of phones sold in 'n<sup>th</sup>' month.

$$\Rightarrow (1.05)^{n-1} > 10$$

$$\Rightarrow \log(1.05^{n-1}) > \log 10.$$

$$(n-1) \log(1.05) > \log(10)$$

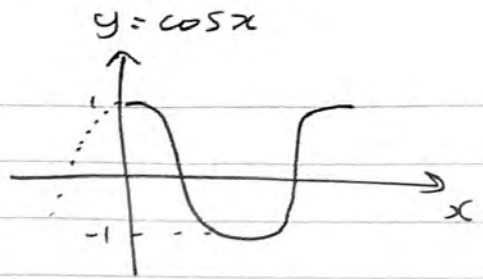
$$\therefore n-1 > \frac{\log 10}{\log 1.05}$$

$$n-1 > 47.194\dots$$

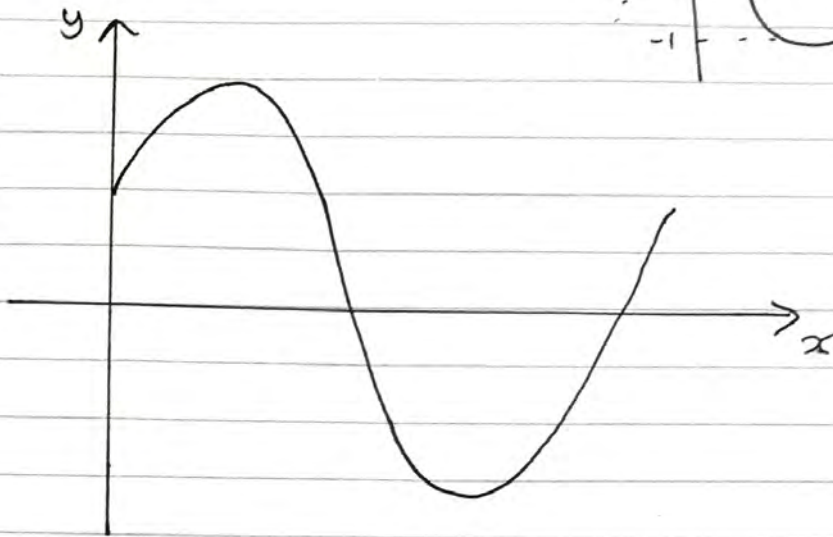
$$\therefore n > 48.194\dots \text{ so } \boxed{N = 49}$$

(next integer)

10a)  $y = \cos\left(x - \frac{\pi}{3}\right)$



TRANSLATION  
IN THE +ve  
x-direction BY  
 $\left(\frac{\pi}{3}\right)$



b) at  $y=0$  :  $\cos\left(x - \frac{\pi}{3}\right) = 0$

$$x - \frac{\pi}{3} = \cos^{-1}(0) = \frac{\pi}{2} //, \frac{3\pi}{2} // = x - \frac{\pi}{3}$$

$$\text{so } x = \frac{\pi}{2} + \frac{\pi}{3}, \frac{3\pi}{2} + \frac{\pi}{3}$$

$$x = \frac{5\pi}{6}, \frac{11\pi}{6}$$

$$\text{so } \boxed{\left(\frac{5\pi}{6}, 0\right)} \text{ and } \boxed{\left(\frac{11\pi}{6}, 0\right)}$$

$$\text{at } x=0 : y = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \therefore \boxed{\left(0, \frac{1}{2}\right)}$$

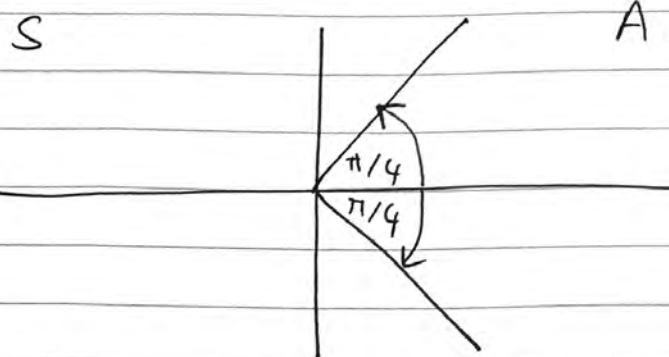
c)  $0 \leq x \leq 2\pi$

$$\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x - \frac{\pi}{3} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} //$$

new interval :  $-\frac{\pi}{3} \leq x - \frac{\pi}{3} \leq \frac{5\pi}{3}$

Using CAST :



$$x - \frac{\pi}{3} = \frac{\pi}{4}, -\frac{\pi}{4},$$

$$\therefore x = \frac{\pi}{4} + \frac{\pi}{3}, -\frac{\pi}{4} + \frac{\pi}{3}$$

$$x = \boxed{\frac{7\pi}{12}}, \boxed{\frac{\pi}{12}}$$

11a)

$a$	$a + d$	$a + 2d$
$60$	$4p$	$2p - 6$

$$a + d = 4p \quad \text{--- (1)}$$

$$a + 2d = 2p - 6 \quad \text{--- (2)}$$

$$\textcircled{2} - \textcircled{1} : a + 2d - a - d = 2p - 6 - 4p$$

$$\therefore d = -2p - 6 //$$

substitute into (1) :  $a - 2p - 6 = 4p$

$$(a = 60) \quad 60 - 6 = \overset{6p}{\cancel{2p}} = 54$$

$$\therefore p = \frac{54}{6} = \boxed{9}$$

$$b) n^{\text{th}} \text{ term} = a + (n-1)d$$

$$20^{\text{th}} \text{ term} = 60 + 19d$$

$$\text{from (a), } d = -2p - 6 = -2(9) - 6 = \underline{\underline{-24}}$$

$$\therefore 20^{\text{th}} \text{ term} = 60 + 19(-24) = \boxed{-396}$$

$$c) S_n = \frac{n}{2} [2a + (n-1)d]$$

$$\left. \begin{array}{l} d = -24 \\ a = 60 \end{array} \right\} S_n = \frac{n}{2} [120 + (n-1)(-24)]$$

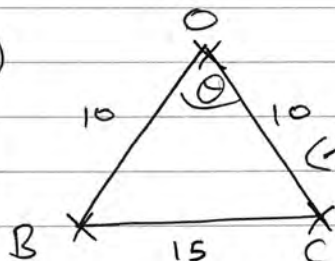
$$= \frac{n}{2} [120 + 24(1-n)]$$

compare with given result to obtain required expression:

$$S_n = \frac{n}{2} [120 + 24 - 24n] = \frac{n}{2} [144 - 24n]$$

$$= 12n \left[ \frac{144}{24} - n \right] = \underline{\underline{12n[6-n]}}$$

12a)



cosine rule

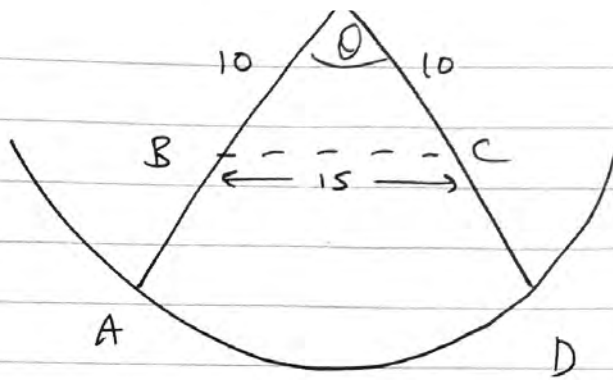
$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore \cos \theta = \frac{10^2 + 10^2 - 15^2}{2(10)(10)} = \frac{-1}{8}$$

$$\therefore \theta = \cos^{-1}\left(-\frac{1}{8}\right) = \boxed{1.696^c}$$

b)



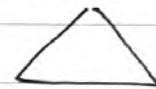
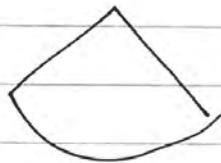
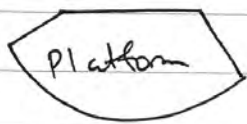
$$\theta = 1.696^\circ$$

$$AB = CD = 22 - 10 = \underline{\underline{12 \text{ m}}}$$

$$AD \text{ length} = r\theta = 22 \times 1.696 \approx 37.31$$

$$\therefore \text{Perimeter} = 12 + 12 + 15 + 37.31 = \boxed{76.31}$$

$$c) \text{ Area } ABCD = \text{OAD} - \text{OBC}$$



$$\begin{aligned} \text{Area} &= \frac{1}{2} (22^2) (1.696) - \left( \frac{1}{2} \right) (10^2) (\sin(1.696)) \\ &= 410.432 - 49.6 \\ &= \boxed{361 \text{ m}^2} \end{aligned}$$

$$13a) y = \frac{(x-3)(3x-25)}{x} = \frac{3x^2 - 34x + 75}{x}$$

$$y = 3x - 34 + \frac{75}{x} = 3x - 34 + 75x^{-1}$$

$$\therefore \frac{dy}{dx} = 3 - 75x^{-2} = \boxed{3 - \frac{75}{x^2}}$$

at turning point,  $\frac{dy}{dx} = 0$ .

$$3 - \frac{75}{x^2} = 0$$

$$3 = \frac{75}{x^2}$$

$$3x^2 = 75 \longrightarrow x^2 = 25 //$$

$$\therefore x = \sqrt{25} = 5 \quad (x \neq -5, x > 0)$$

reject  $x = -5$ .

$$\text{at } x = 5, y = \frac{(5-3)(3(5)-25)}{5} = -4 //$$

$$\therefore \boxed{(5, -4)}$$

c)  $\frac{dy}{dx} = 3 - 75x^{-2}$

$$\frac{d^2y}{dx^2} = 150x^{-3}$$

$$\text{at } x = 5, \frac{d^2y}{dx^2} = \frac{150}{5^3} > 0$$

$\therefore$  it's a minimum.

$$d) P\left(\frac{5}{2}, y\right)$$

substituting  $x = \frac{5}{2}$  to find  $y$ ,

$$y = \frac{\left(\frac{5}{2} - 3\right) \left(3\left(\frac{5}{2}\right) - 25\right)}{\frac{5}{2}} = \frac{7}{2} //$$

$$\therefore P\left(\frac{5}{2}, \frac{7}{2}\right)$$

$$\frac{dy}{dx} = 3 - \frac{75}{x^2}$$

$$\hookrightarrow \text{at } x = \frac{5}{2}, \frac{dy}{dx} = -9 // \text{ (at P)}$$

$$\therefore \text{at normal, } m = \frac{1}{9} \quad \left(\frac{1}{9}x - 9 = -1\right)$$

$$y - \frac{7}{2} = \frac{1}{9}\left(x - \frac{5}{2}\right)$$

$$y = \frac{1}{9}x - \frac{5}{18} + \frac{7}{2}$$

$$y = \frac{1}{9}x + \frac{29}{9}$$

$$\Rightarrow \boxed{x - 9y + 29 = 0}$$

14a)  $x^2 - 2x - 15 = 2x - 3$

$$x^2 - 4x - 12 = 0$$

$$(x-6)(x+2) = 0$$

$$x = 6, x = -2.$$

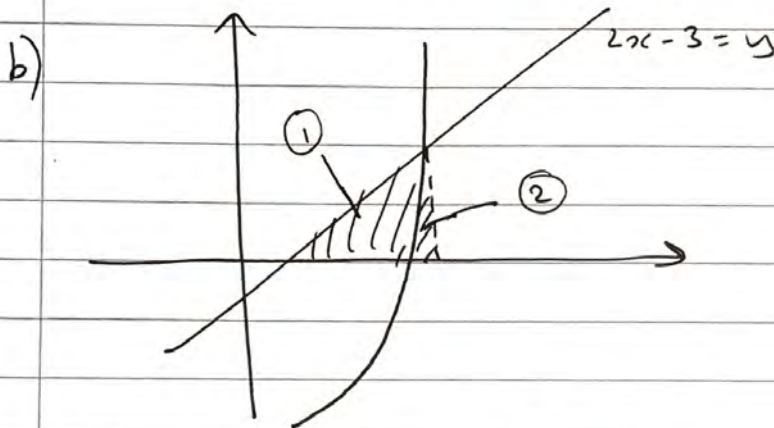


$$y = 2(6) - 3 = 9$$

$$y = 2(-2) - 3 = -7$$

∴ points are: A (-2, -7) B (6, 9)

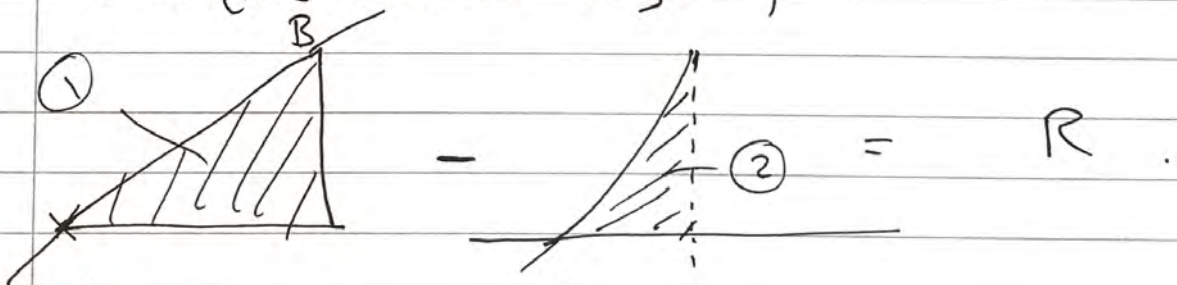
(A is below x-axis).



line  
 $y=0, x = \frac{3}{2} //$

curve  
 $y=0, (x-5)(x+3)=0$   
 $x=5 //$

To acquire shaded region,



$$= \int_{\frac{3}{2}}^6 (2x-3) dx - \int_5^6 (x^2 - 2x - 15) dx = R$$

(OR AREA Δ)

$$R = \left[ x^2 - 3x \right]_{\frac{3}{2}}^6 - \left[ \frac{x^3}{3} - x^2 - 15x \right]_5^6$$

$$= \frac{81}{4} - \frac{13}{3} = \boxed{\frac{191}{12}}$$

15a)  $X(0, 3)$   $Y(6, 11)$

$$m_{xy} = \frac{11-3}{6-0} = \frac{8}{6} = \boxed{\frac{4}{3}}$$

b)  $M$  = midpoint of  $XY$ .

$$\therefore M \left( \frac{0+6}{2}, \frac{11+3}{2} \right) \Rightarrow M(3, 7) //$$

Line  $MZ$  is perpendicular to  $XY$  so gradient will be  $-\frac{3}{4} //$

$$\left( -\frac{3}{4} \times \frac{4}{3} = -1 \right)$$

$$\Rightarrow y - 7 = -\frac{3}{4}(x - 3)$$

$$\Rightarrow y = -\frac{3}{4}x + \frac{9}{4} + 7$$

$$\Rightarrow y = -\frac{3}{4}x + \frac{37}{4}$$

$$\Rightarrow \boxed{4y + 3x - 37 = 0}$$

c) using eqn from (b).

$$y = 10 : 40 + 3x - 37 = 0$$

$$3x = -3$$

$$x = -1 //$$

d)  $Z(-1, 10)$  is the centre.

We now need the radius.

$$\text{radius} = ZX \quad (\text{or } ZY)$$

$$= \sqrt{(-1-0)^2 + (10-3)^2} = \underline{\underline{5\sqrt{2}}}$$

$$\therefore r^2 = 50$$

hence eqn of circle :  $\underline{\underline{(x+1)^2 + (y-10)^2 = 50}}$

expanding :  $x^2 + 2x + 1 + y^2 - 20y + 100 = 50$

$$x^2 + y^2 + 2x - 20y + 51 = 0$$