

1. A sequence of numbers u_1, u_2, u_3, \dots satisfies

$$u_{n+1} = 2u_n - 6, \quad n \geq 1$$

Given that $u_1 = 2$

- (a) find the value of u_3

(2)

- (b) evaluate $\sum_{i=1}^4 u_i$

(3)

$$a) u_2 = 2(2) - 6 = -2$$

$$u_3 = 2(-2) - 6 = -10$$

$$b) \sum_{i=1}^4 u_i = u_1 + u_2 + u_3 + u_4$$

$$= 2 + (-2) + (-10) + [2(-10) - 6]$$

$$= -36$$

2. (i) Given that $\frac{49}{\sqrt{7}} = 7^a$, find the value of a .

(2)

(ii) Show that $\frac{10}{\sqrt{18} - 4} = 5\sqrt{2} + 20$

You must show all stages of your working.

(3)

$$i) \quad 7^a = \frac{49}{\sqrt{7}} = \frac{7^2}{7^{1/2}} = 7^{3/2}$$

$$\Rightarrow a = \frac{3}{2}$$

$$ii) \quad \frac{10}{\sqrt{18} - 4} = \frac{10}{\sqrt{18} - 4} \times \frac{-\sqrt{18} - 4}{-\sqrt{18} - 4}$$

$$= \frac{-10\sqrt{18} - 40}{16 - 18}$$

$$= 5\sqrt{18} + 20$$

3. Find, using calculus and showing each step of your working,

$$\int_1^4 \left(6x - 3 - \frac{2}{\sqrt{x}} \right) dx$$

$$\begin{aligned} \int_1^4 (6x - 3 - 2x^{-1/2}) dx &= [3x^2 - 3x - 4x^{1/2}]_1^4 \quad (5) \\ &= [3(4)^2 - 3(4) - 4(4)^{1/2}] - [3(1)^2 - 3(1) - 4(1)^{1/2}] \\ &= 48 - 12 - 8 - 3 + 3 + 4 \\ &= 32 \end{aligned}$$

4. The 4th term of an arithmetic sequence is 3 and the sum of the first 6 terms is 27

Find the first term and the common difference of this sequence.

$$u_n = a + (n-1)d$$

$$S_n = \frac{n}{2} [2a + (n-1)d] \quad (6)$$

$$3 = a + (4-1)d$$

$$27 = \frac{6}{2} [2a + (6-1)d]$$

$$a = 3 - 3d \quad (1)$$

$$27 = 6a + 15d \quad (2)$$

$$(1) \text{ in } (2): 27 = 6(3 - 3d) + 15d$$

$$= 18 - 3d$$

$$d = -3$$

$$\text{In } (1): a = 3 - 3(-3)$$

$$= 12$$

5. (a) Sketch the graph of $y = \sin 2x$, $0 \leq x \leq \frac{3\pi}{2}$

Show the coordinates of the points where your graph crosses the x -axis.

(2)

The table below gives corresponding values of x and y , for $y = \sin 2x$.

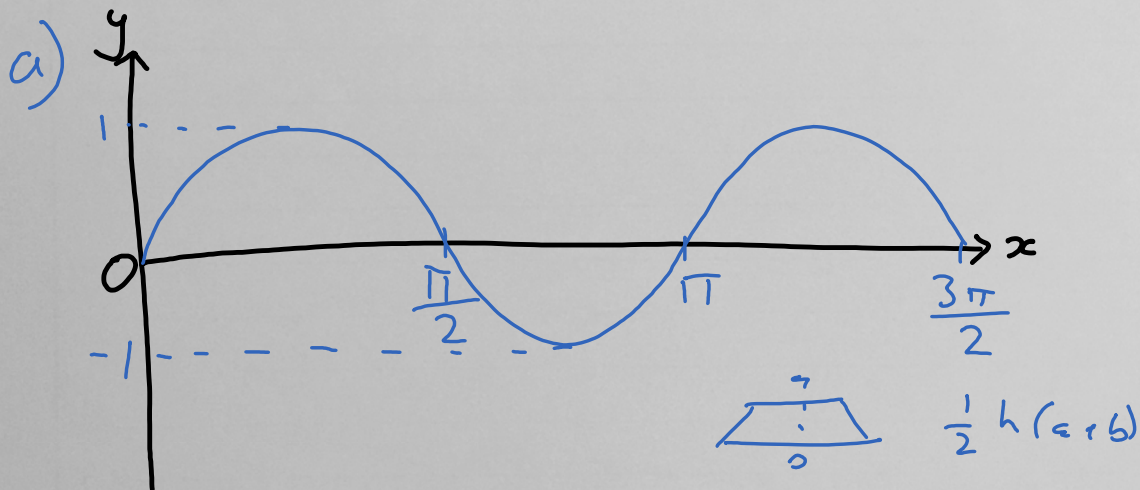
The values of y are rounded to 3 decimal places where necessary.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$
y	0	0.5	0.866	1

(b) Use the trapezium rule with all the values of y from the table to find an approximate value for

$$\int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

(3)



b)

$$\int_0^{\pi/4} \sin 2x \, dx \approx \frac{1}{2} \cdot \frac{\pi}{12} [0 + 1 + 2(0.5 + 0.866)]$$

$$= 0.489 \text{ (3dp)}$$

6. Mathematics · 2016 · Jan · Paper $f(x) = x^3 + x^2 - 12x - 18$

(a) Use the factor theorem to show that $(x + 3)$ is a factor of $f(x)$.

(2)

(b) Factorise $f(x)$.

(2)

(c) Hence find exact values for all the solutions of the equation $f(x) = 0$

(3)

$$\begin{aligned} \text{a) } f(-3) &= (-3)^3 + (-3)^2 - 12(-3) - 18 \\ &= -27 + 9 + 36 - 18 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{b) } f(x) &= x^3 + x^2 - 12x - 18 \\ &= (x + 3)(x^2 - 2x - 6) \end{aligned}$$

$$\text{c) } f(x) = 0$$

$$\begin{aligned} x &= -3 \quad \text{or} \quad x = \frac{2 \pm \sqrt{(-2)^2 + 4(6)}}{2} \\ &= 1 \pm \sqrt{7} \end{aligned}$$

7. (a) Find the first 4 terms, in ascending powers of x , of the binomial expansion of $(1 + kx)^8$, where k is a non-zero constant. Give each term in its simplest form.

(4)

Given that the coefficient of x^3 in this expansion is 1512

(b) find the value of k .

(3)

$$\begin{aligned}
 \text{a) } (1 + kx)^8 &\approx 1 + 8kx + 8(7)(kx)^2 \frac{1}{2} \\
 &\quad + 8(7)(6)(kx)^3 \frac{1}{3!} \\
 &= 1 + 8kx + 28k^2x^2 + 56k^3x^3
 \end{aligned}$$

$$\text{b) } 1512 = 56k^3$$

$$k^3 = 27$$

$$k = 3$$

8. (a) Given that $7 \sin x = 3 \cos x$, find the exact value of $\tan x$.

(1)

(b) Hence solve for $0 \leq \theta < 360^\circ$

$$7 \sin(2\theta + 30^\circ) = 3 \cos(2\theta + 30^\circ)$$

giving your answers to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

$$a) 7 \sin x = 3 \cos x$$

$$\tan x = \frac{3}{7}$$

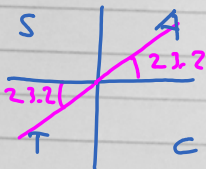
$$b) 7 \sin(2\theta + 30) = 3 \cos(2\theta + 30)$$

$$\tan(2\theta + 30) = \frac{3}{7}$$

$$0 \leq \theta < 360$$

$$30 \leq 2\theta + 30 < 750$$

$$2\theta + 30 = \cancel{23.2}, 203.2, \\ 383.2, 563.2, \\ 743.2$$



$$\theta = 86.6, 176.6, 266.6, 356.6$$

9. The resident population of a city is 130 000 at the end of Year 1

A model predicts that the resident population of the city will increase by 2% each year, with the populations at the end of each year forming a geometric sequence.

(a) Show that the predicted resident population at the end of Year 2 is 132 600

(1)

(b) Write down the value of the common ratio of the geometric sequence.

(1)

The model predicts that Year N will be the first year which will end with the resident population of the city exceeding 260 000

(c) Show that

$$N > \frac{\log_{10} 2}{\log_{10} 1.02} + 1$$

(4)

(d) Find the value of N .

(1)

$$\begin{aligned} \text{a) } P_2 &= 1.02 P_1 = 1.02 \times 130000 \\ &= 132600 \end{aligned}$$

$$\text{b) } 1.02$$

$$\text{c) } P_N = P_1 (1.02)^{N-1}$$

$$130000 (1.02)^{N-1} > 260000$$

$$1.02^{N-1} > 2$$

$$\log_{10} 1.02^{N-1} > \log_{10} 2$$

$$(N-1) \log_{10} 1.02 > \log_{10} 2$$

$$N > \frac{\log_{10} 2}{\log_{10} 1.02} + 1$$

10. The curve C has equation

$$y = 12x^{\frac{5}{4}} - \frac{5}{18}x^2 - 1000, \quad x > 0$$

(a) Find $\frac{dy}{dx}$

(2)

(b) Hence find the coordinates of the stationary point on C .

(5)

(c) Use $\frac{d^2y}{dx^2}$ to determine the nature of this stationary point.

(3)

$$a) \frac{dy}{dx} = 15x^{1/4} - \frac{5x}{9}$$

$$c) \frac{d^2y}{dx^2} = \frac{15}{4}x^{-3/4} - \frac{5}{9}$$

$$b) 0 = 15x^{1/4} - \frac{5x}{9}$$

When $x = 81$,

$$= 27x^{1/4} - x$$

$$\frac{d^2y}{dx^2} = \frac{1}{4} \cdot \frac{15}{81^{3/4}} - \frac{5}{9}$$

$$= x^{1/4} (27 - x^{3/4})$$

$$= -\frac{5}{12} < 0$$

$$x = 0 \quad \text{or} \quad x^{3/4} = 27$$

\therefore Max pt.

But $x > 0$

$$x = 27^{4/3}$$

$$= 81$$

$$\text{When } x = 81, y = 12(81)^{5/4} - \frac{5}{18}(81)^2 - 1000$$

$$= 93.5$$

\therefore Stationary point at $(81, 93.5)$

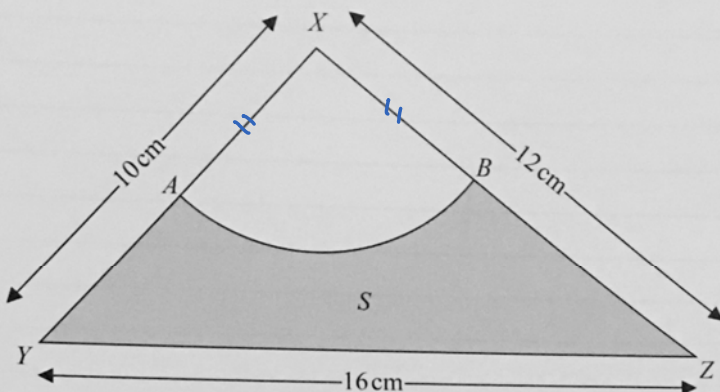


Figure 1

Figure 1 shows a triangle XYZ with $XY = 10$ cm, $YZ = 16$ cm and $ZX = 12$ cm.

- (a) Find the size of the angle YXZ , giving your answer in radians to 3 significant figures.

(3)

The point A lies on the line XY and the point B lies on the line XZ and $AX = BX = 5$ cm. AB is the arc of a circle with centre X .

The shaded region S , shown in Figure 1, is bounded by the lines BZ , ZY , YA and the arc AB .

Find

- (b) the perimeter of the shaded region to 3 significant figures,

(4)

- (c) the area of the shaded region to 3 significant figures.

(4)

$$a) \cos \widehat{YXZ} = \frac{12^2 + 10^2 - 16^2}{2 \times 10 \times 12}$$

$$\widehat{YXZ} = 1.62^\circ \text{ (3 sf)}$$

$$b) \widehat{AB} = 5 \times 1.62 = 8.10 \text{ cm}$$

$$P = 8.10 + (10 - 5) + (12 - 5) + 16 = 36.1 \text{ cm}$$

$$c) \text{Area } XYZ = \frac{1}{2} \times 10 \times 12 \sin 1.62 = 59.9 \text{ cm}^2$$

$$\text{Area of sector} = \frac{1}{2} \times 5^2 \times 1.62 = 20.3 \text{ cm}^2$$

$$S = 59.9 - 20.3 = 39.7 \text{ cm}^2$$

12.

$$f(x) = \frac{(4 + 3\sqrt{x})^2}{x}, \quad x > 0$$

(a) Show that $f(x) = Ax^{-1} + Bx^k + C$, where A , B , C and k are constants to be determined.

(4)

(b) Hence find $f'(x)$.

(2)

(c) Find an equation of the tangent to the curve $y = f(x)$ at the point where $x = 4$

(4)

$$a) f(x) = \frac{16 + 24\sqrt{x} + 9x}{x} = 16x^{-1} + 24x^{-1/2} + 9$$

$$b) f'(x) = -16x^{-2} - 12x^{-3/2}$$

$$c) f'(4) = -\frac{16}{4^2} - \frac{12}{4^{3/2}} = -1 - \frac{3}{2} = -\frac{5}{2}$$

$$f(4) = \frac{(4 + 3\sqrt{4})^2}{4} = 25$$

$$y - y_1 = m(x - x_1)$$

$$y - 25 = -\frac{5}{2}(x - 4)$$

$$y = 35 - \frac{5}{2}x$$

13. The equation $k(3x^2 + 8x + 9) = 2 - 6x$, where k is a real constant, has no real roots.

(a) Show that k satisfies the inequality

$$11k^2 - 30k - 9 > 0$$

(4)

(b) Find the range of possible values for k .

(4)

$$a) k(3x^2 + 8x + 9) = 2 - 6x$$

$$3kx^2 + 8kx + 9k = 2 - 6x$$

$$3kx^2 + (8k + 6)x + 9k - 2 = 0$$

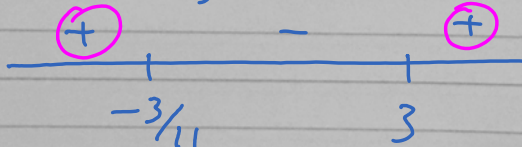
$$\Delta = (8k + 6)^2 - 4(3k)(9k - 2) < 0$$

$$64k^2 + 96k + 36 - 108k^2 + 24k < 0$$

$$44k^2 - 120k - 36 > 0$$

$$11k^2 - 30k - 9 > 0$$

$$b) (11k + 3)(k - 3) > 0$$



$$\left\{ k < -\frac{3}{11} \right\} \cup \left\{ k > 3 \right\}$$

14. (i) Given that

$$\log_a x + \log_a 3 = \log_a 27 - 1, \text{ where } a \text{ is a positive constant}$$

find, in its simplest form, an expression for x in terms of a .

(4)

(ii) Solve the equation

$$(\log_5 y)^2 - 7(\log_5 y) + 12 = 0$$

showing each step of your working.

(4)

$$\text{i) } \log_a x + \log_a 3 = \log_a 27 - 1$$

$$\log_a \left(\frac{3x}{27} \right) = -1$$

$$\frac{x}{9} = a^{-1}$$

$$x = \frac{9}{a}$$

$$\text{ii) } (\log_5 y)^2 - 7(\log_5 y) + 12 = 0$$

$$\text{Let } x = \log_5 y:$$

$$x^2 - 7x + 12 = 0$$

$$(x - 4)(x - 3) = 0$$

$$\log_5 y = 4 \quad \text{or} \quad \log_5 y = 3$$

$$y = 5^4$$

$$= 625$$

$$y = 5^3$$

$$= 125$$

15. The points A and B have coordinates $(-8, -8)$ and $(12, 2)$ respectively. AB is the diameter of a circle C .

(a) Find an equation for the circle C .

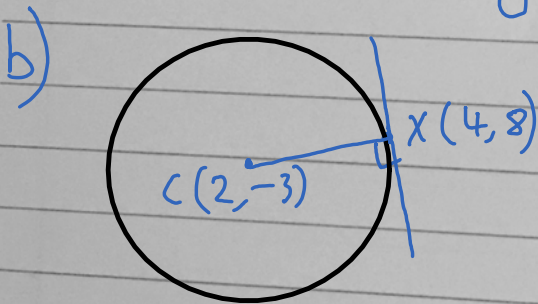
The point $(4, 8)$ also lies on C .

(b) Find an equation of the tangent to C at the point $(4, 8)$, giving your answer in the form $ax + by + c = 0$

a) Centre is mid-pt of AB : $C(2, -3)$

$$\text{Radius, } r^2 = (12 - 2)^2 + (2 + 3)^2 = 125$$

$$\Rightarrow (x - 2)^2 + (y + 3)^2 = 125$$



$$m_{CX} = \frac{8 + 3}{4 - 2} = \frac{11}{2}$$

$$m_{\text{tangent}} = -\frac{2}{11}$$

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -\frac{2}{11}(x - 4)$$

$$2x + 11y - 96 = 0$$

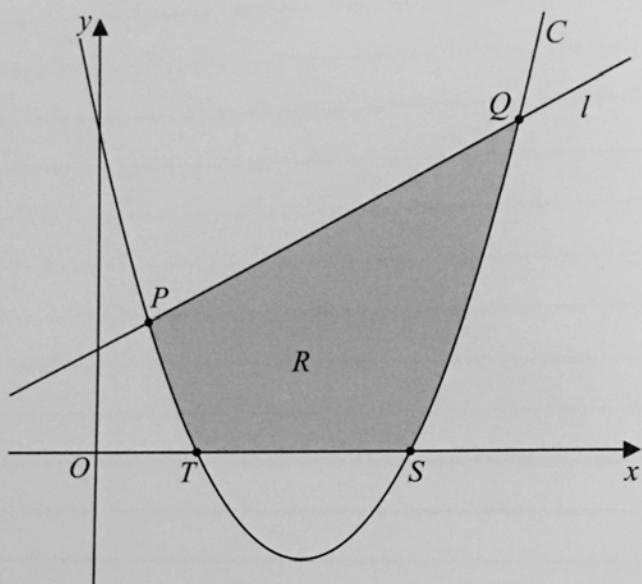


Figure 2

The straight line l with equation $y = \frac{1}{2}x + 1$ cuts the curve C , with equation $y = x^2 - 4x + 3$, at the points P and Q , as shown in Figure 2

(a) Use algebra to find the coordinates of the points P and Q .

(5)

The curve C crosses the x -axis at the points T and S .

(b) Write down the coordinates of the points T and S .

(2)

The finite region R is shown shaded in Figure 2. This region R is bounded by the line segment PQ , the line segment TS , and the arcs PT and SQ of the curve.

(c) Use integration to find the exact area of the shaded region R .

$$a) \frac{1}{2}x + 1 = x^2 - 4x + 3$$

(8)

$$0 = 2x^2 - 9x + 4$$

$$= (2x - 1)(x - 4)$$

$$x = \frac{1}{2} \text{ or } 4$$

$$\text{When } x = \frac{1}{2}, y = \frac{1}{2}\left(\frac{1}{2}\right) + 1 = \frac{5}{4}$$

$$x = 4, y = \frac{1}{2}(4) + 1 = 3$$

$$\therefore P\left(\frac{1}{2}, \frac{5}{4}\right) \text{ and } Q(4, 3)$$

$$b) y = x^2 - 4x + 3$$

$$= (x-3)(x-1)$$

$$\therefore T(1,0) \text{ and } S(3,0)$$

$$c) R = \int_{1/2}^4 \left[\frac{x}{2} + 1 - (x^2 - 4x + 3) \right] dx - \left| \int_1^3 (x^2 - 4x + 3) dx \right|$$

$$= \int_{1/2}^4 \left(-x^2 + \frac{9x}{2} - 2 \right) dx + \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$= \left[-\frac{x^3}{3} + \frac{9x^2}{4} - 2x \right]_{1/2}^4 + 9 - 18 + 9 - \frac{1}{3} + 2 - 3$$

$$= -\frac{64}{3} + 36 - 8 + \frac{1}{24} - \frac{9}{16} + 1 - \frac{4}{3}$$

$$= \frac{93}{16}$$