C12 Jan 2017 (MA)

Q1a) $y = \frac{x^3}{3} - 2x^2 + 3x + 5$

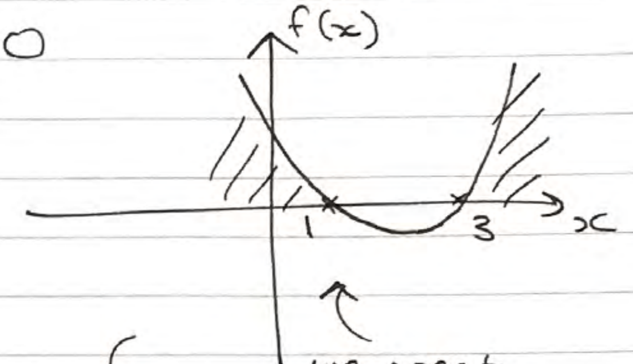
$$\frac{dy}{dx} = \boxed{x^2 - 4x + 3}$$

b) $x^2 - 4x + 3 > 0$

Solving $[x^2 - 4x + 3 = 0]$ to find critical values:

$\Rightarrow (x-3)(x-1) = 0$

$\therefore x = 3, \ x = 1$



we want where $\frac{dy}{dx} > 0$

(ie where $y > 0$ on the graph)

$\therefore$ set of values:

$$\boxed{\begin{array}{l} x > 3 \\ \text{and} \quad x < 1 \end{array}}$$

2a) $x^2 + y^2 - 8x + 4y - 12 = 0$

$(x-4)^2 - 16 + (y+2)^2 - 4 - 12 = 0$

$\therefore (x-4)^2 + (y+2)^2 = 32$

$\therefore C \boxed{(4, -2)}$

b) $r = \sqrt{32} = \boxed{4\sqrt{2}}$

b) at y-axis, $\underline{x=0}$: $(0-4)^2+(y+2)^2=32$

$$(y+2)^2 = 16$$

$$y+2 = \pm\sqrt{16}$$

$$y+2 = \pm 4$$

$$\therefore \quad y = -2 \pm 4$$

so $\underline{y=-6}$ and $\underline{y=2}$

$\therefore$ A $(0, 2)$ and B $(0, -6)$

(It doesn't matter which point is labelled A or B)

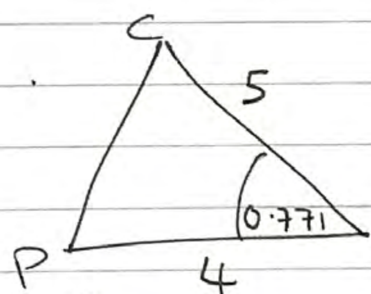
3a) $l = r\theta = 7 \times 0.8 = \boxed{5.6\text{cm}}$

b) $\angle$ AOC will be 90° $(\frac{\pi}{2})$ as OABC is a rectangle.

$\therefore$ $\angle POC = 180^\circ - 90^\circ - \angle AOQ$

in radians: $\angle POC = \pi - \frac{\pi}{2} - 0.8 = \boxed{0.771^c}$

c) we require the length CP.



cosine rule

$CP^2 = 5^2 + 4^2 - 2(5)(4)\cos(0.771)$

$CP^2 = 12.311\ldots$ $\therefore$ $CP = 3.5088\ldots$

$\therefore$ Perimeter $= 3.5088 + 4 + 7 + 5.6 + 5 + 7 = \boxed{32.11\text{ cm}}$

4a) $a + (a+d) + (a+2d) + \ldots$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_9 = \frac{9}{2}\left[2a + (8)d\right] = 54$

$12 = 2a + 8d$

$\div 2$: $\underline{6 = a + 4d}$

b) 8^{th} term $= a + 7d$

7^{th} term $= a + 6d$.

8^{th} term is half the 7^{th} term

$\therefore a + 7d = \frac{1}{2}(a + 6d)$.

$2a + 14d = a + 6d$

$a = -8d$ — ②

from (a), $a = 6 - 4d$

→ ② : $6 - 4d = -8d$

$-4d = 6$

$\boxed{d = -\frac{3}{2}}$ $\therefore a = -8\left(-\frac{3}{2}\right)$

$\boxed{a = 12}$

5ai) $y = \log_3 x$

$\log_3\left(\frac{x}{9}\right) = \log_3(x) - \log_3(9)$

$= \boxed{y - 2}$

ii) $\log_3 \sqrt{x} = \log_3(x^{\frac{1}{2}}) = \frac{1}{2}\log_3 x$

$= \boxed{\frac{1}{2} y}$

b) $2(y-2) - (\frac{1}{2}y) = 2$

$2y - 4 - \frac{y}{2} = 2$

$\frac{3y}{2} = 6 \quad \therefore \quad y = \frac{6}{\frac{3}{2}} = \boxed{4}$

substitute back into $[y = \log_3 x]$ to find x

$4 = \log_3 x$.

$3^4 = \boxed{x = 81}$

Remember we are solving for x not y

6ai) $2y = 3x + 5$

$y = \frac{3}{2}x + \frac{5}{2}$

$\therefore$ gradient$_{l_1}$ $= \boxed{\frac{3}{2}}$

ii) $y=0$: $3x+5=0$ $\therefore$ $\boxed{x = -\frac{5}{3}}$

b) l_2 is perp. to l_1 $\therefore$ $m_{l_2} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$ //

put $x=1$ into l_1 to find y: $2y = 8$ $\therefore y = 4$ //

so l_2 passes through $(1, 4)$.

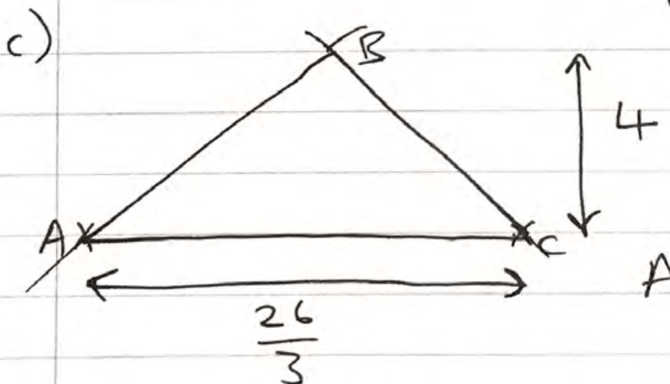
$\Rightarrow$ $y - 4 = -\frac{2}{3}(x - 1)$

$y = -\frac{2}{3}x + \frac{2}{3} + 4$

$y = -\frac{2}{3}x + \frac{14}{3}$

$\times 3$: $3y = -2x + 14$

$\boxed{2x + 3y - 14 = 0}$

c) 

$y=0$: $x=7$ //

$\therefore C\ (7, 0)$

so $AC = 7 + \frac{5}{3} = \frac{26}{3}$

Area $= \frac{1}{2} \times \frac{26}{3} \times 4$

$= \boxed{\frac{52}{3}}$

7 i) $\int\left(\frac{2+4x^3}{x^2}\right)dx = \int\left[2x^{-2}+4x\right]dx$

$= \left[\frac{2x^{-1}}{-1} + \frac{4x^2}{2}\right] + c$

$= \boxed{-\frac{2}{x} + 2x^2 + c}$

ii) $\int_2^4 \left[4x^{-\frac{1}{2}} + k\right]dx = 30$

$\Rightarrow \left[\frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + kx\right]_2^4 = 30$

$\Rightarrow \left[8\sqrt{x} + kx\right]_2^4 = 30$

$\Rightarrow \left[16 + 4k\right] - \left[8\sqrt{2} + 2k\right] = 30$

$\Rightarrow 16 - 8\sqrt{2} + 2k = 30$

$\Rightarrow 2k = 14 + 8\sqrt{2}$

$\therefore \boxed{k = 7 + 4\sqrt{2}}$

8a) $f(x) = 2x^3 - 5x^2 - 23x - 10$

Long Division

$$
\begin{array}{r l}
 & 2x^2 + x - 20 \\
x-3 & \overline{)\,2x^3 - 5x^2 - 23x - 10} \\
 & \underline{2x^3 - 6x^2} \\
 & 0 \quad x^2 - 23x \\
 & \underline{x^2 - 3x} \\
 & 0 \quad -20x - 10 \\
 & \underline{-20x + 60} \\
 & 0 \quad -70
\end{array}
$$

$\leftarrow$ remainder $= \boxed{-70}$

b) if $(x+2)$ is a factor then $f(-2) = 0$.

$f(-2) = 2(-2)^3 - 5(-2)^2 - 23(-2) - 10$

$= -36 + 46 - 10 = 0$

$\therefore (x+2)$ is a factor.

c)

$$
\begin{array}{r l}
 & 2x^2 - 9x - 5 \\
x+2 & \overline{)\,2x^3 - 5x^2 - 23x - 10} \\
 & \underline{2x^3 + 4x^2} \\
 & 0 \quad -9x^2 - 23x \\
 & \underline{-9x^2 - 18x} \\
 & 0 \quad -5x - 10 \\
 & \underline{-5x - 10} \\
 & 0 \quad 0
\end{array}
$$

$\therefore f(x) = (x+2)(2x^2 - 9x - 5)$

$2x^2 - 9x - 5 = (2x+1)(x-5)$

$\therefore f(x) = \boxed{(x+2)(2x+1)(x-5)}$

d) $2(3^{3t}) - 5(3^{2t}) - 23(3^{t}) = 10$

let $y = 3^t$

$\Rightarrow 2y^3 - 5y^2 - 23y = 10$

$\Rightarrow 2y^3 - 5y^2 - 23y - 10 = 0$ — this is exactly what we factorised in (c).

$\Rightarrow (y+2)(2y+1)(y-5) = 0$ //

$\therefore$ $y = -2 \longrightarrow 3^t = -2$ no valid solutions. (REJECT).

$2y+1=0$: $y = -\frac{1}{2} \longrightarrow 3^t = -\frac{1}{2}$ no valid solutions (REJECT).

$y-5=0$: $y = 5 \longrightarrow 3^t = 5$

$3^t = 5$

$\log(3^t) = \log(5) = t\log(3)$

$\therefore t = \dfrac{\log 5}{\log 3} = \boxed{1.465}$

9a) $f(x) = 8x^{-1} + \frac{1}{2}x - 5$

$f'(x) = -8x^{-2} + \frac{1}{2} = 0$ $\left(\frac{dy}{dx} = 0 \text{ at } A\right)$.

$\Rightarrow \quad \frac{1}{2} = \frac{8}{x^2}$.

$\Rightarrow \quad x^2 = 16$. $\therefore x = 4$ (rej. $x = -4$).
($x > 0$)

$f(4) = \frac{8}{4} + 2 - 5 = -1$

$\therefore \boxed{A(4,-1)}$

b i) $x = 2$, $x = 8$ [no change to x]

ii) $\boxed{(4,1)}$ [vertical translation of $+2$]

ii) x-coordinates are multiplied by $\frac{1}{4}$

$\therefore$ roots: $x = \frac{1}{2}$, $x = 2$

10a) $(1+ax)^{20} = (1)^{20} + \binom{20}{1}(1)^{19}(ax)^{1} + \binom{20}{2}(1)^{18}(ax)^{2}$

$= 1 + 20ax + 190a^2x^2.$

Compare with given terms to obtain:

$20a = 4 \quad — \ (1)$

$190a^2 = p \quad — \ (2)$

from (1): $a = \frac{4}{20} = \boxed{\frac{1}{5}}$

b) from (2): $p = 190a^2 = 190\left(\frac{1}{5}\right)^2 = \boxed{\frac{38}{5}} = p$

c) 4^{th} term $\longrightarrow x^3$ term.
5^{th} term $\longrightarrow x^4$ term.

5^{th} term $= \binom{20}{4}(1)^{16}(ax)^{4}$

$= (4845a^4)x^4$

$\therefore q = 4845a^4 = 4845\left(\frac{1}{5}\right)^4 = \boxed{\frac{969}{125}}$

11i) $\underline{0 \leq x < 2\pi}$

$$3\cos^2 x + 1 = 4\sin^2 x$$

$$3 - 3\sin^2 x + 1 = 4\sin^2 x$$

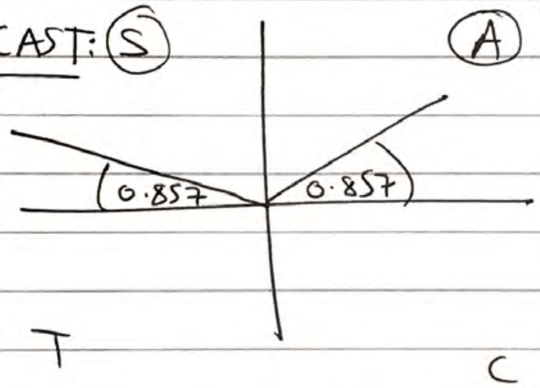
$$7\sin^2 x = 4.$$

$$\sin^2 x = \frac{4}{7}$$

$$\therefore \sin x = \pm \frac{2}{\sqrt{7}}$$

$$x = \sin^{-1}\left(\frac{2}{\sqrt{7}}\right) = 0.857^c$$

By CAST:

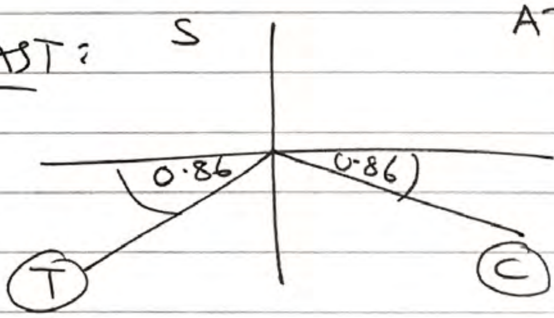


$x = 0.86, \pi - 0.86$

$\boxed{x = 0.86^c, 2.28^c}$

and $x = \sin^{-1}\left(-\frac{2}{\sqrt{7}}\right) = -0.857^c$

By CAST:



$x = \pi + 0.86^c,$
$2\pi - 0.86^c$

$\boxed{x = 4.00^c, 5.42^c}$

ii) $\underline{0 \leq \theta < 360}$

$$5\sin(\theta + 10) = \cos(\theta + 10)$$

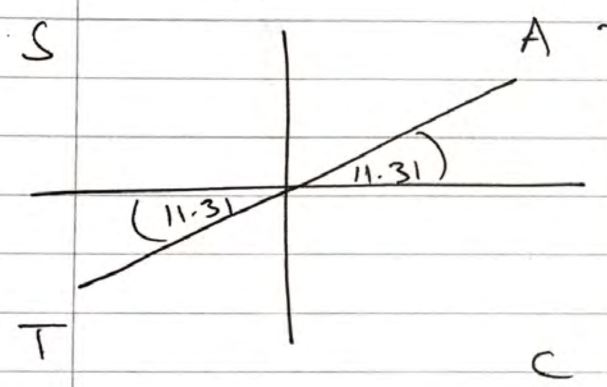
$\boxed{\div \cos(\theta+10)}$: $5\tan(\theta + 10^\circ) = 1$

$$\tan(\theta + 10^\circ) = \frac{1}{5}$$

$$\theta + 10^\circ = \tan^{-1}\left(\frac{1}{5}\right) = 11 \cdot 31^\circ$$

$\underline{\text{new range}}$: $\boxed{10 \leq \theta + 10^\circ < 370^\circ}$

<u>By CAST</u>



$$\theta + 10 = 11 \cdot 31^\circ, 191 \cdot 31^\circ$$

$\therefore \boxed{\theta = 1 \cdot 3^\circ, 181 \cdot 3^\circ}$

12a) $y = \frac{3}{4}x^2 - 4x^{\frac{1}{2}} + 7$ $\quad P(4, 11)$.

$$\frac{dy}{dx} = \frac{3}{2}x - 2x^{-\frac{1}{2}}$$

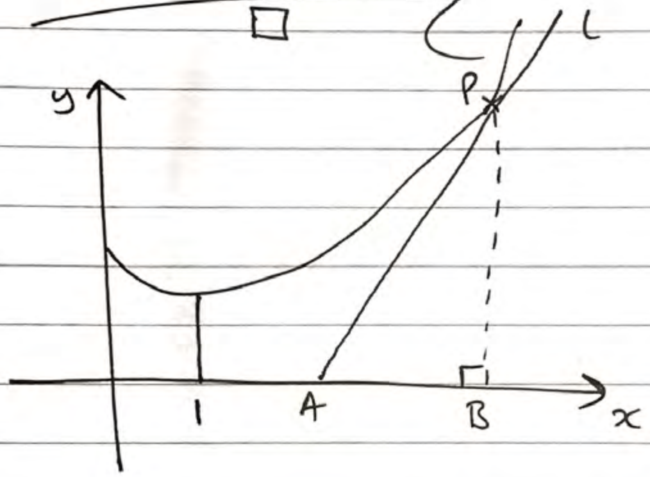
at P, $\frac{dy}{dx} = \frac{3}{2}(4) - \frac{2}{\sqrt{4}} = 5$

$\therefore y - 11 = 5(x - 4)$ ← l.

$y = 5x - 20 + 11$

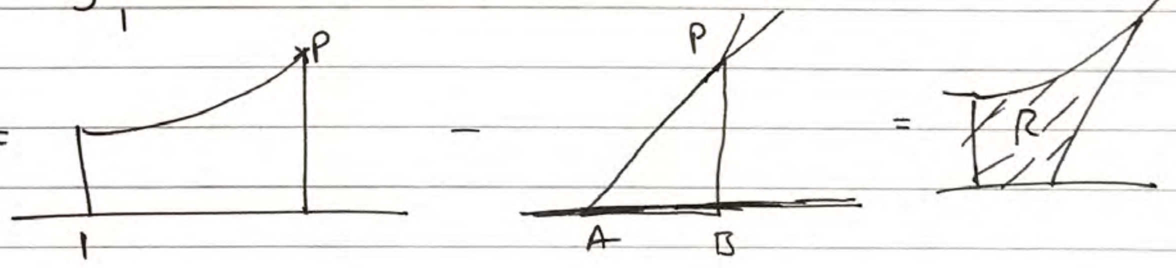
$y = 5x - 9$ $\square$

b)



$$R = \int_1^4 [y]\,dx - \text{Area}_{ABP}.$$

R =



Area $\Delta = \frac{1}{2}(AB)(11)$

to find length AB . . .

put $y = 0$ into l : $5x - 9 = 0$

$x = 9/5$

$\therefore AB = 4 - \frac{9}{5} = 11/5$

$\therefore$ Area $\Delta = \frac{1}{2} \times \frac{11}{5} \times 11 = \boxed{\frac{121}{10}}$

$$R = \int_1^4 \left[\frac{3x^2}{4} - 4x^{\frac{1}{2}} + 7\right] dx - \frac{121}{10}$$

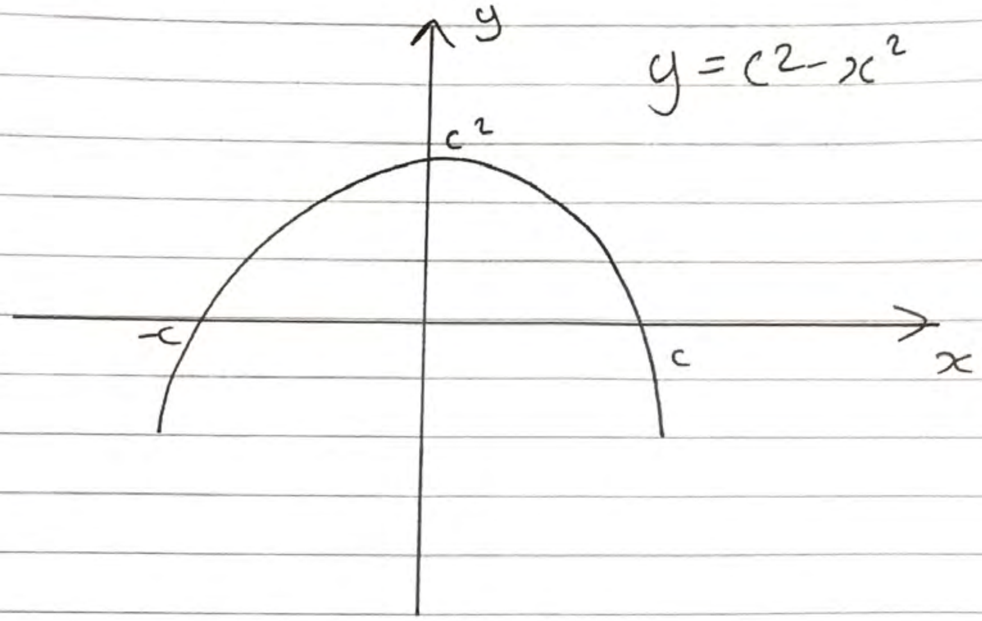
$$= \left[\frac{x^3}{4} - \frac{4x^{3/2}}{\frac{3}{2}} + 7x\right]_1^4 - \frac{121}{10}$$

$$= \left[16 - \frac{64}{3} + 28\right] - \left[\frac{55}{12}\right] - \frac{121}{10}$$

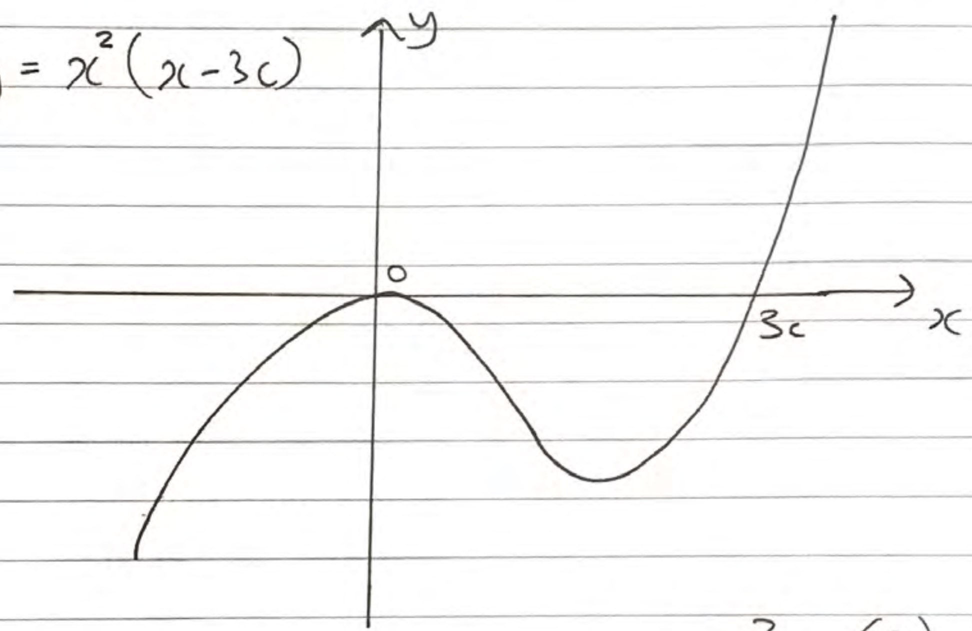
$$= \boxed{\frac{359}{60}} = \underline{5.98} \ (2d.p).$$

GradeMax

13a i)



ii) $y = x^2(x - 3c)$



$y = x^3 - (3c)x^2$

$x = 0$: $y = 0$.

$x = 3c$: $y = 0$.

general cubic shape.

b) $c^2 - x^2 = x^2(x - 3c)$.

$c^2 - x^2 = x^3 - 3cx^2$.

$x^3 + x^2 - 3cx^2 - c^2 = 0$

$x^3 + (1-3c)x^2 - c^2 = 0$.

$\square$

c) $x = 2$ is a solution.... $\therefore$ substitute $x = 2$ into eqn. from (b).

$\Rightarrow (2)^3 + (1-3c)(2^2) - c^2 = 0$.

$8 + 4(1-3c) - c^2 = 0$.

$12 - 12c - c^2 = 0$

$c^2 + 12c - 12 = 0$

Quadratic formula

$$\left.\begin{array}{l} a=1 \\ b=12 \\ c=-12 \end{array}\right\} \quad c = \frac{-12 \pm \sqrt{144 - 4(-12)}}{2} = \frac{-12 \pm \sqrt{192}}{2}$$

$$c = \frac{-12 + 8\sqrt{3}}{2} = \boxed{-6 + 4\sqrt{3}}$$

$c > 0$ so reject $c = \dfrac{-12 - \sqrt{192}}{2}$

14a) $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ — ①

$rS_n = ar + ar^2 + \ldots + ar^n$ — ②

① − ② : $S_n - rS_n = a - ar^n$

$S_n(1-r) = a(1-r^n)$

$\therefore S_n = \dfrac{a(1-r^n)}{1-r}$

b) $\left.\begin{array}{l} a=180 \\ r=0.93 \end{array}\right\}$ end of 5^{th} year $= ar^5 = 180 \times (0.93)^5$

$= \underline{\underline{125.2}}$.

c) $\underset{a}{180} + \underset{ar}{180(0.93)} + \underset{ar^2}{180(0.93)^2} + \ldots + \underset{ar^{20}}{180(0.93)^{20}}$

start of 1^{st} year — end of 1^{st} year — end of 2^{nd} — end of 20^{th} year.

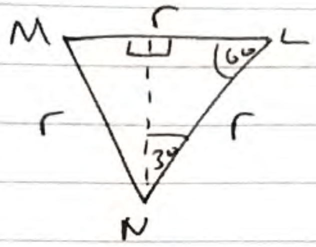
Because we want the total liquid at the end of 20 years, use:

$\left.\begin{array}{l} a = 180(0.93) \\ r = 0.93 \end{array}\right\} \quad S_{20} = \dfrac{180(0.93)(1-0.93^{20})}{1-0.93}$

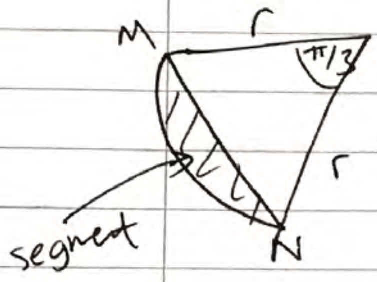
$= \boxed{1831 \text{ litres}}$

15) ΔLMN is also equilateral.

$$\text{Area}_{LMN} = \frac{1}{2} \times r \times r\cos 30^\circ = \frac{r^2\sqrt{3}}{4}$$ //



Now to find the area of one of the segments outside the ΔLMN:



$\therefore$ Area of sector MLN $= \frac{1}{2}(r^2)\left(\frac{\pi}{3}\right)$

$$= \frac{\pi r^2}{6}$$ //

so $\text{area}_{\text{segment}} = -\frac{r^2\sqrt{3}}{4} + \frac{r^2\pi}{6}$

$$= r^2\left(-\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)$$ //

$$\therefore R = 3 \times r^2\left(-\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right) + \frac{r^2\sqrt{3}}{4}$$

$$= r^2\left(-\frac{3\sqrt{3}}{4} + \frac{\pi}{2} + \frac{\sqrt{3}}{4}\right) = \boxed{r^2\left(-\frac{\sqrt{3}}{2} + \frac{\pi}{2}\right)}$$

$$= \boxed{r^2\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2}\right)}$$