C2 Jan 2014 (IAL) (MA)

Q1) $(1+px)^{12} = (1)^{12} + \binom{12}{1}(1)^{11}\cdot(px)^1 + \binom{12}{2}(1)^{10}(px)^2$

$(1+px)^{12} = 1 + 12px + 66p^2x^2$ //

$\therefore 18 = 12p$

$\Rightarrow p = \frac{18}{12} = \boxed{\frac{3}{2}}$

$66p^2 = q = 66\left(\frac{3}{2}\right)^2 = \boxed{\frac{297}{2} = q}$

2a) $f(x) = 2x^3 + x^2 + ax + b$

$f(2) = 25$: $f(2) = 2(8) + 4 + 2a + b = 25$

$\Rightarrow 20 + 2a + b = 25$

$\Rightarrow 2a + b = 5$ //

$\square$

b)

$$\begin{array}{r l}
 & 2x^2 - 5x + (a+15) \\
x+3 & \overline{)\,2x^3 + x^2 + ax + b} \\
 & 2x^3 + 6x^2 \\
 & 0 \quad -5x^2 + ax \\
 & \quad -5x^2 - 15x \\
 & \quad\quad 0 \quad (a+15)x + b \\
 & \quad\quad (a+15)x + 3a + 45 \\
 & \quad\quad 0 \quad b - 3a - 45
\end{array}$$

$b - 3a - 45$ // ← remainder

so because $(x+3)$ is a factor, the remainder should be 0.

$\therefore \quad b - 3a - 45 = 0$ //

now solve simultaneously with eqn. from a,

from a, $b = 5 - 2a$

$\therefore\ 5 - 2a - 3a = 45$

$-5a = 40 \quad \therefore \boxed{a = -8}$

so $b = 5 - 2(-8) = \boxed{21 = b}$

<u>alt</u>: $(x+3)$ is a facter so $f(-3) = 0$.
sub $x = -3$ into $f(x)$ and equate to 0.

3ai) $y = 2x^{\frac{1}{2}} + 18x^{-\frac{1}{2}} - 1$

$\frac{dy}{dx} = x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}$

ii) $\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{27}{2}x^{-\frac{5}{2}}$

b) $\frac{dy}{dx} = 0$: $\frac{1}{x^{\frac{1}{2}}} - \frac{9}{x^{3/2}} = 0$

$\Rightarrow \quad \frac{1}{x^{\frac{1}{2}}} = \frac{9}{x^{\frac{3}{2}}}$

taking reciprocal of both sides : $x^{\frac{1}{2}} = \frac{1}{9}x^{\frac{3}{2}}$.

$\Rightarrow x^{\frac{1}{2}} - \frac{1}{9}x^{\frac{3}{2}} = 0$

$\Rightarrow x^{\frac{1}{2}}\left(1 - \frac{1}{9}x\right) = 0$ //

[$x > 0 \ \therefore$ reject $x = 0$]

$1 - \frac{x}{9} = 0 \Rightarrow \boxed{x = 9}$

at $x = 9$, $y = 2\sqrt{9} + \frac{18}{\sqrt{9}} - 1 = 11$.

$\therefore$ point is $\boxed{(9, 11)}$.

c) from (aii), $\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}}{x^{3/2}} + \frac{27}{2x^{5/2}}$

at $x = 9$, $\frac{d^2y}{dx^2} = \frac{-1}{54} + \frac{1}{18} = \frac{1}{27} > 0$ //

$\therefore$ its a minimum point.

4ai) $\left.\begin{array}{l} a = 5 \\ r = 1.2 \end{array}\right\}$ 20^{th} term $= ar^{19} = 5 \times 1.2^{19} = \boxed{159.7}$

ii) $S_{20} = \frac{5(1-(1.2^{20}))}{1-1.2} = \boxed{933.4}$

b) $S_n = \frac{5(1-(1.2)^n)}{1-1.2} > 3000$

$\frac{5(1-(1.2)^n)}{-0.2} > 3000$

$(\times -0.2)$ $5(1-(1.2)^n) < -600$] when x by a -ve number the signs flip.

$\Rightarrow$ $1-(1.2)^n < -120$

$\Rightarrow$ $1.2^n > 121$

$$\log(1.20^n) > \log(121)$$

$$n\log(1.2) > \log(121)$$

$$n > \frac{\log(121)}{\log(1.2)}$$

$$n > 26.3\ldots$$

$$\therefore \boxed{n_{min} = 27}$$

5a) $\underline{t=1}$: $H = 10 + 5\sin\frac{\pi}{6} = 10 + \frac{5}{2} = \boxed{12.5m}$

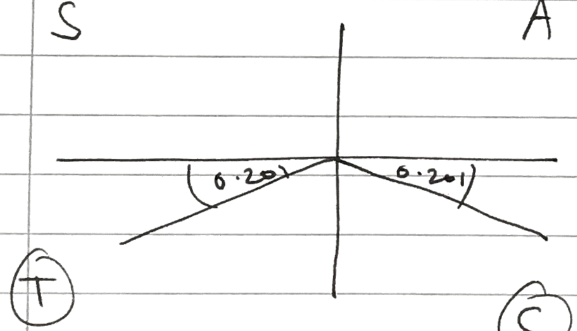
b) $\underline{H=9}$: $9 = 10 + 5\sin\left(\frac{\pi t}{6}\right)$

$$-\frac{1}{5} = \sin\left(\frac{\pi t}{6}\right).$$

$$\therefore \sin^{-1}\left(-\frac{1}{5}\right) = \frac{\pi t}{6} = \ldots -0.20^c$$

Before midday means that $t < 12$.

$\therefore$ our range is : $0 \leq \frac{\pi t}{6} < 2\pi$



$$\frac{\pi t}{6} = \pi + 0.20^c, (2\pi - 0.20)^c$$

$$\frac{\pi t}{6} = 3.34, 6.08$$

$$t = 6.38, 11.61$$

6.38 hours $\longrightarrow$ $\boxed{06:23}$

11.61 hours $\longrightarrow$ $\boxed{11:37}$ AM.

6) $\log_x(7y+1) - \log_x(2y) = 1.$

$\log_x\left(\frac{7y+1}{2y}\right) = 1$

$x^1 = \frac{7y+1}{2y}$

$2xy = 7y+1$

$2xy - 7y = 1$

$y(2x-7) = 1$

$\boxed{y = \frac{1}{2x-7}}$

7a) $\frac{dy}{dx} = 3x^2 - 12x + 9$

at P, $\frac{dy}{dx} = \quad 3(4^2) - 12(4) + 9 = 9$

$\therefore$ gradient of normal at P $= -\frac{1}{9}$

$\left(-\frac{1}{9} \times 9 = -1\right).$

$\Rightarrow \quad y - 9 = \frac{-1}{9}(x-4)$

$\Rightarrow \quad y = -\frac{1}{9}x + \frac{4}{9} + 9$

$\Rightarrow \quad y = \frac{-1}{9}x + \frac{85}{9}.$

$\times 9$: $9y = -x + 85 \quad \therefore \boxed{x + 9y = 85}$

b) $R = \int_0^4 (y_2 - y_1)\,dx$

$$= \int_0^4 \left[-\frac{1}{9}x + \frac{85}{9} - x^3 + 6x^2 - 9x - 5\right] dx$$

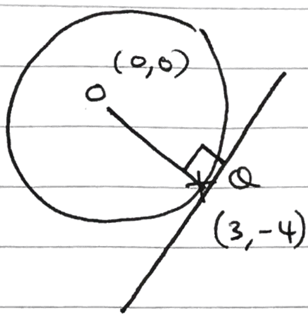
$$= \int_0^4 \left[-x^3 + 6x^2 - \frac{82}{9}x + \frac{40}{9}\right] dx$$

$$= \left[\frac{-x^4}{4} + 2x^3 - \frac{82x^2}{18} + \frac{40}{9}x\right]_0^4$$

$$= \left[-64 + 128 - \frac{656}{9} + \frac{160}{9}\right] = \boxed{\frac{80}{9}}$$

8a) $x^2 + y^2 = 25.$

b)



<u>Finding gradient of line OQ</u>

$$M_{OQ} = \frac{-4-0}{3-0} = -\frac{4}{3}$$

$\therefore$ gradient of tangent $= \frac{3}{4}$

(tangent at Q will be $\perp$ to OQ).

$$y+4 = \frac{3}{4}(x-3)$$

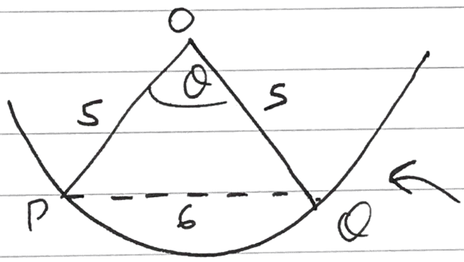
$$y = \frac{3}{4}x - \frac{9}{4} - 4$$

$$y = \frac{3}{4}x - \frac{25}{4}$$

$\times 4$: $4y = 3x - 25$

$$\boxed{3x - 4y = 25}$$

c)



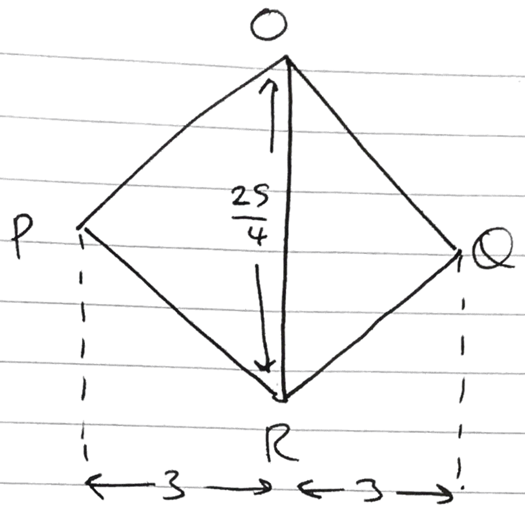
length PQ $= 3 - -3 = 6$

<u>cosine rule</u>

$$\cos POQ = \frac{5^2+5^2-6^2}{2(5)(5)} = \frac{7}{25}$$

$$\therefore \angle POQ = \cos^{-1}\left(\frac{7}{25}\right) = \boxed{1.287^c}$$

d)



using eqn from (b),

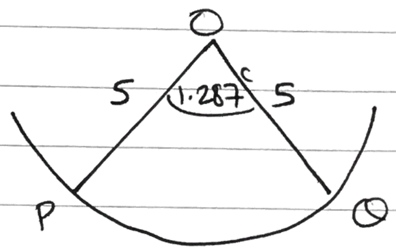
$3x - 4y = 25$

$\underline{x=0}$; $y = -25/4$

$\therefore |OR| = \dfrac{25}{4}$ //

$\therefore \text{Area}_{OPR} = \text{Area}_{OQR} = \frac{1}{2} \times \frac{25}{4} \times 3 = \frac{75}{8}$.

$\therefore \text{Area}_{OPRQ} = 2 \times \frac{75}{8} = \frac{75}{4}$ //



$\text{Area}_{OPQ} = \frac{1}{2}(5^2)(1.287^c)$

$= 16.0875$ //

Area required $= \frac{75}{4} - 16.0875 = \boxed{2.66 \text{ units}^2}$

9a) $5\sin x - (1 - \sin^2 x) + 2\sin^2 x = 1$

$3\sin^2 x + 5\sin x - 1 = 1$

$3\sin^2 x + 5\sin x - 2 = 0$

b) $3\sin^2 2\theta + 5\sin 2\theta - 2 = 0$

$\hookrightarrow (3\sin 2\theta - 1)(\sin 2\theta + 2) = 0$

$3\sin 2\theta - 1 = 0 \rightarrow \sin 2\theta = \frac{1}{3}$ //

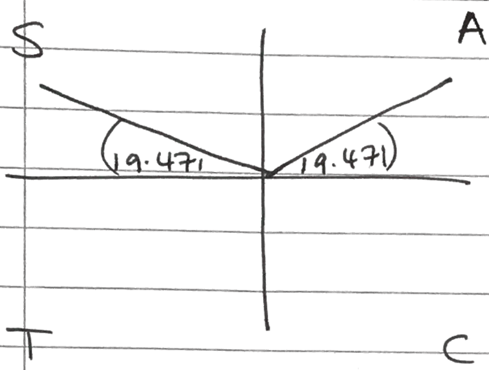
$2\theta = \sin^{-1}\left(\frac{1}{3}\right) = 19.471^\circ$ //

$\sin 2\theta + 2 = 0$

$\sin 2\theta = -2$ ✗

reject: no valid solutions

Solving in: $\boxed{-360 \leq 2\theta < 360}$



$2\theta = 19.471°, 180 - 19.471°, -(180 + 19.471°), -(360 - 19.471°)$

$2\theta = 19.47°, 160.53°, -199.47°, 340.53°$

$$\boxed{\theta = 9.74°, 80.26°, -99.74°, -170.26°}$$