Leave blank

1. The curve C has equation

$$y = \frac{3x-2}{(x-2)^2}, \quad x \neq 2$$

The point P on C has x coordinate 3

Find an equation of the normal to C at the point P in the form $ax + by + c = 0$, where a, b and c are integers.

(6)

When $x = 3$, $y = \frac{3(3)-2}{(3-2)^2} = 7$

$$\frac{dy}{dx} = \frac{(x-2)^2 \cdot 3 - 2(x-2)(3x-2)}{(x-2)^4}$$

When $x = 3$, $\frac{dy}{dx} = \frac{(3-2)^2 \cdot 3 - 2(3-2)(9-2)}{(3-2)^4}$

$= -11$

$\therefore m = \frac{1}{11}$

$y - y_1 = m(x - x_1)$

$y - 7 = \frac{1}{11}(x - 3)$

$11y - 77 = x - 3$

$x - 11y + 74 = 0$

Leave blank

2. Solve, for $0 \leqslant \theta < 2\pi$,

$$2\cos 2\theta = 5 - 13\sin\theta$$

Give your answers in radians to 3 decimal places.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

$2(1 - 2\sin^2\theta) = 5 - 13\sin\theta$

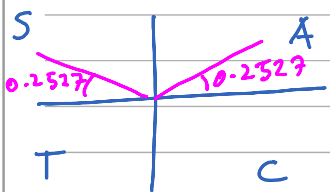
$2 - 4\sin^2\theta = 5 - 13\sin\theta$

$4\sin^2\theta - 13\sin\theta + 3 = 0$

$(4\sin\theta - 1)(\sin\theta - 3) = 0$

$\sin\theta = \frac{1}{4}$ or $\sin\theta = 3$

No soln.



$\theta = 0.253,\ 2.889\ (3dp)$

Leave blank

3. The function g is defined by

$$g : x \mapsto |8 - 2x|, \qquad x \in \mathbb{R}, \qquad x \geqslant 0$$

(a) Sketch the graph with equation $y = \mathrm{g}(x)$, showing the coordinates of the points where the graph cuts or meets the axes.

(3)

(b) Solve the equation

$$|8 - 2x| = x + 5$$

(3)

The function f is defined by

$$\mathrm{f} : x \mapsto x^2 - 3x + 1, \qquad x \in \mathbb{R}, \qquad 0 \leqslant x \leqslant 4$$

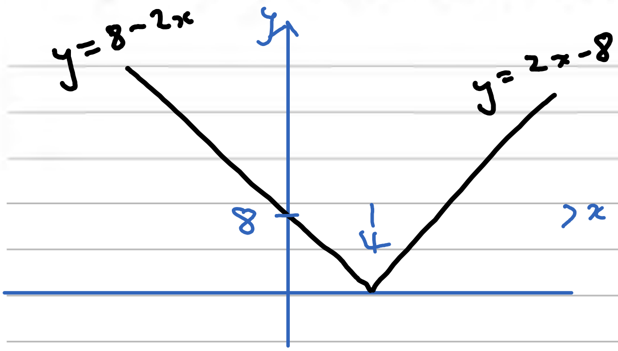
(c) Find fg(5).

(2)

(d) Find the range of f. You must make your method clear.

(4)

a)



$0 = 8 - 2x$

$x = 4$

$g(0) = 8$

b) $8 - 2x = x + 5$ OR $-(8 - 2x) = x + 5$

$3 = 3x$ $\qquad x = 13$

$x = 1$

c) $f(x) = x^2 - 3x + 1$, $g(x) = |8 - 2x|$

$fg(5) = f[|8 - 2(5)|] = f(2)$

$= (2)^2 - 3(2) + 1$

$= -1$

Leave blank

Question 3 continued

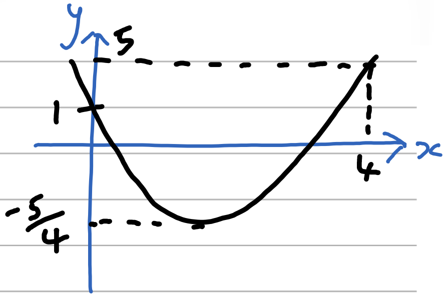
d) $f(x) = x^2 - 3x + 1$

$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$

$f(0) = (0)^2 - 3(0) + 1 = 1$

$f(4) = (4)^2 - 3(4) + 1 = 5$

$\Rightarrow -\frac{5}{4} \leqslant f(x) \leqslant 5$



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4. Use the substitution $x = 2\sin\theta$ to find the exact value of

$$\int_0^{\sqrt{3}} \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$$

(7)

$x = 2\sin\theta$ When $x = 0$, $0 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ $\theta = 0$

$x = \sqrt{3}$, $\sqrt{3} = 2\sin\theta$

$\theta = \pi/3$

$$\int_0^{\sqrt{3}} \frac{1}{(4-x^2)^{3/2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{2\cos\theta\, d\theta}{(4-4\sin^2\theta)^{3/2}}$$

$$= \int_0^{\pi/3} \frac{2\cos\theta}{(4)^{3/2}(1-\sin^2\theta)^{3/2}}\,d\theta$$

$$= \int_0^{\pi/3} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta$$

$$= \int_0^{\pi/3} \frac{1}{4\cos^2\theta}\,d\theta$$

$$= \int_0^{\pi/3} \frac{1}{4}\sec^2\theta\, d\theta$$

$$= \left[\frac{1}{4}\tan\theta\right]_0^{\pi/3}$$

$$= \frac{1}{4}\left(\sqrt{3} - 0\right)$$

$$= \frac{\sqrt{3}}{4}$$

Leave blank

5. (a) Use the binomial expansion, in ascending powers of x, of $\frac{1}{\sqrt{(1-2x)}}$ to show that

$$\frac{2+3x}{\sqrt{(1-2x)}} \approx 2 + 5x + 6x^2, \qquad |x| < 0.5$$

(4)

(b) Substitute $x = \frac{1}{20}$ into

$$\frac{2+3x}{\sqrt{(1-2x)}} = 2 + 5x + 6x^2$$

to obtain an approximation to $\sqrt{10}$

Give your answer as a fraction in its simplest form.

(3)

a) $\frac{1}{\sqrt{1-2x}} = (1-2x)^{-1/2}$

$\approx 1 + \left(-\frac{1}{2}\right)(-2x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-2x)^2 \cdot \frac{1}{2}$

$= 1 + x + \frac{3x^2}{2}$

$\frac{2+3x}{\sqrt{1-2x}} \approx (2+3x)\left(1 + x + \frac{3x^2}{2}\right)$

$= 2 + 5x + 6x^2$

b) $\frac{2+3(1/20)}{\sqrt{1-2(1/20)}} = 2 + 5\left(\frac{1}{20}\right) + 6\left(\frac{1}{20}\right)^2$

$\frac{43}{20} \times \sqrt{\frac{10}{9}} = 2 + \frac{5}{20} + \frac{3}{200}$

$\frac{43}{60}\sqrt{10} = \frac{453}{200}$

$\sqrt{10} = \frac{1359}{430}$

Leave blank

6. (i) Given $x = \tan^2 4y$, $0 < y < \frac{\pi}{8}$, find $\frac{dy}{dx}$ as a function of x.

Write your answer in the form $\frac{1}{A(x^p + x^q)}$, where A, p and q are constants to be found.

(5)

(ii) The volume V of a cube is increasing at a constant rate of 2 cm³ s⁻¹. Find the rate at which the length of the edge of the cube is increasing when the volume of the cube is 64 cm³.

(5)

i) $x = \tan^2 4y$

$$\frac{dx}{dy} = 2\tan 4y\,(\sec^2 4y)\cdot 4$$

$$= 8\tan 4y\,(1+\tan^2 4y)$$

$$= 8(\tan 4y + \tan^3 4y)$$

$$= 8(x^{1/2} + x^{3/2})$$

$$\frac{dy}{dx} = \frac{1}{8(x^{1/2}+x^{3/2})}$$

ii) $V = x^3$, $\frac{dV}{dt} = 2\text{cms}^{-1}$, find $\frac{dx}{dt}$

$$\frac{dV}{dx} = 3x^2$$

$$\frac{dx}{dt} = \frac{dx}{dV}\cdot\frac{dV}{dt} = \frac{2}{3x^2}$$

When $V = 64$, $x = 4$

$$\frac{dx}{dt} = \frac{2}{3(4)^2} = \frac{1}{24}\text{cms}^{-1}$$

Leave blank

7. (a) Given that

$$2\cos(x + 30)^\circ = \sin(x - 30)^\circ$$

without using a calculator, show that

$$\tan x^\circ = 3\sqrt{3} - 4$$

(5)

(b) Hence or otherwise solve, for $0 \leqslant \theta < 180$,

$$2\cos(2\theta + 40)^\circ = \sin(2\theta - 20)^\circ$$

Give your answers to one decimal place.

(4)

a) $2\cos(x+30) = \sin(x-30)$

$2\cos x\cos 30 - 2\sin x\sin 30 = \sin x\cos 30 - \cos x\sin 30$

$\sqrt{3}\cos x - \sin x = \frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x$

$2\sqrt{3} - 2\tan x = \sqrt{3}\tan x - 1$

$(\sqrt{3}+2)\tan x = 2\sqrt{3}+1$

$\tan x = \frac{2\sqrt{3}+1}{\sqrt{3}+2}$

$= \frac{(2\sqrt{3}+1)(\sqrt{3}-2)}{3-4}$

$= \frac{6-4\sqrt{3}+\sqrt{3}-2}{-1}$

$= 3\sqrt{3}-4$

b) $x+30 = 2\theta+40$ $\qquad x = 2\theta+10$

$\theta = \frac{x-10}{2}$ $\qquad 0 \leqslant \theta < 180$

$10 \leqslant x < 370$

Leave blank

Question 7 continued

$\tan x = 3\sqrt{3} - 4$

$x = 50.1, 230.1$

$\theta = \dfrac{x - 10}{2}$

$= 20.1, 110.1$

Leave blank

8.

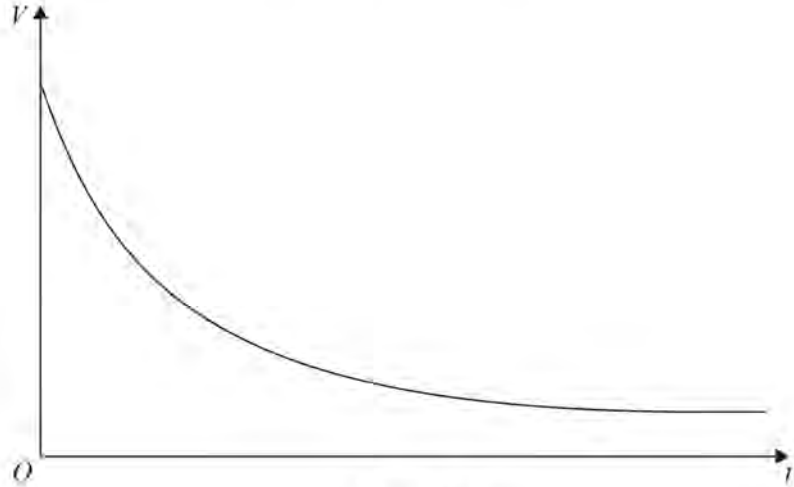

Figure 1

The value of Lin's car is modelled by the formula

$$V = 18\,000e^{-0.2t} + 4000e^{-0.1t} + 1000, \quad t \geqslant 0$$

where the value of the car is V pounds when the age of the car is t years.

A sketch of t against V is shown in Figure 1.

(a) State the range of V.

(2)

According to this model,

(b) find the rate at which the value of the car is decreasing when $t = 10$
Give your answer in pounds per year.

(3)

(c) Calculate the exact value of t when $V = 15000$

(4)

a) At $t=0$, $V = 18000 + 4000 + 1000 = 23000$

As $t \to \infty$, $V \to 1000$

$\therefore 1000 < V \leq 23000$

b) $\left.\dfrac{dV}{dt}\right|_{t=10} = -0.2(18000)e^{-0.2(10)} - 0.1(4000)e^{-0.1(10)}$

$= -£634/\text{year}$

Leave blank

Question 8 continued

c) $15000 = 18000e^{-0.2t} + 4000e^{-0.1t} + 1000$

$0 = 18(e^{-0.1t})^2 + 4e^{-0.1t} - 14$

$= 9(e^{-0.1t})^2 + 2(e^{-0.1t}) - 7$

$= (9e^{-0.1t} - 7)(e^{-0.1t} + 1)$

$e^{-0.1t} = \frac{7}{9}$ or $e^{-0.1t} = -1$

No soln.

$-0.1t = \ln\left(\frac{7}{9}\right)$

$t = 10\ln\left(\frac{9}{7}\right)$

Leave blank

9.

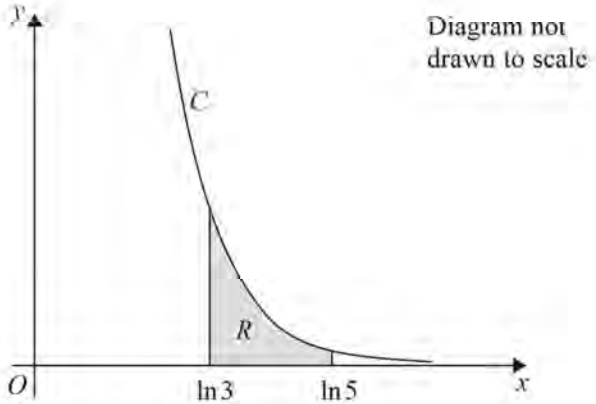

Figure 2

The curve C has parametric equations

$$x = \ln(t+2), \quad y = \frac{4}{t^2} \qquad t > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve C, the x-axis and the lines with equations $x = \ln 3$ and $x = \ln 5$

(a) Show that the area of R is given by the integral

$$\int_1^3 \frac{4}{t^2(t+2)}\,dt$$

(3)

(b) Hence find an exact value for the area of R.

Write your answer in the form $(a + \ln b)$, where a and b are rational numbers.

(7)

(c) Find a cartesian equation of the curve C in the form $y = f(x)$.

(2)

a) when $x = \ln 3$, $\ln 3 = \ln(t+2)$

$t = 1$

$x = \ln 5$, $\ln 5 = \ln(t+2)$

$t = 3$

$\frac{dx}{dt} = \frac{1}{t+2}$

$R = \int_{\ln 3}^{\ln 5} y\,dx = \int_1^3 y\frac{dx}{dt}\,dt = \int_1^3 \frac{4}{t^2}\cdot\frac{1}{t+2}\,dt$

Leave blank

Question 9 continued

b) $$\frac{4}{t^2(t+2)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+2}$$

$$At(t+2) + B(t+2) + Ct^2 = 4$$

$t=-2: \ 4C = 4$ $\quad t=0: \ 2B = 4$

$C = 1$ $\quad B = 2$

$t=1: \ 1(3)A + 3B + C = 4$

$$A = \frac{4-3(2)-1}{3} = -1$$

$$\therefore R = \int_1^3 \left(\frac{1}{t+2} + \frac{2}{t^2} - \frac{1}{t}\right) dt$$

$$= \left[\ln|t+2| - \frac{2}{t} - \ln|t|\right]_1^3$$

$$= \ln 5 - \frac{2}{3} - \ln 3 - \ln 3 + 2 + \ln 1$$

$$= \frac{4}{3} + \ln\left(\frac{5}{9}\right)$$

c) $x = \ln(t+2), \quad y = \frac{4}{t^2}$

$e^x - 2 = t$

$$\Rightarrow y = \frac{4}{(e^x-2)^2}$$

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10.

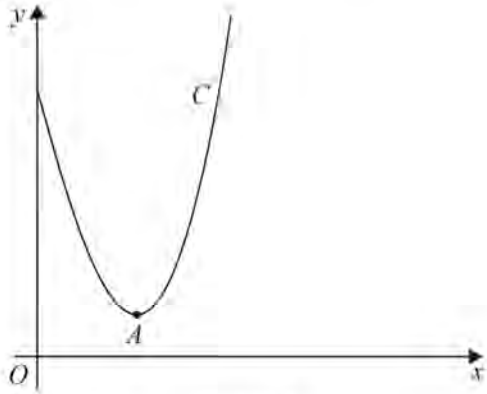


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 4, \quad x > 0$$

Point A is the minimum turning point on the curve.

(a) Show, by using calculus, that the x coordinate of point A is a solution of

$$x = \frac{6}{1 + \ln(x^2)}$$

(5)

(b) Starting with $x_0 = 2.27$, use the iteration

$$x_{n+1} = \frac{6}{1 + \ln(x_n^{\ 2})}$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

(c) Use your answer to part (b) to deduce the coordinates of point A to one decimal place.

(2)

a) $y = \frac{x^2 \ln x}{3} - 2x + 4$

$\frac{dy}{dx} = \frac{x^2}{3} \cdot \frac{1}{x} + \frac{2x}{3} \cdot \ln x - 2 = 0$

$x + 2x \ln x = 6$

$x(1 + 2\ln x) = 6$

$x = \frac{6}{1 + \ln(x^2)}$

Leave blank

Question 10 continued

b) $x_{n+1} = \dfrac{6}{1+\ln(x_n^2)}$, $x_0 = 2.27$

$x_1 = \dfrac{6}{1+\ln(2.27^2)} = 2.273$

$x_2 = \dfrac{6}{1+\ln(2.273^2)} = 2.271$

$x_3 = \dfrac{6}{1+\ln(2.271^2)} = 2.273$

c) When $x = 2.27$,

$y = \dfrac{(2.27)^2 \ln(2.27)}{3} - 2(2.27) + 4$

$= 0.87$

$\therefore A(2.3, 0.9)$

Leave blank

11. With respect to a fixed origin O the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 14 \\ -6 \\ -13 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \qquad l_2: \mathbf{r} = \begin{pmatrix} p \\ -7 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 1 \end{pmatrix}$$

where λ and μ are scalar parameters and p and q are constants.

Given that l_1 and l_2 are perpendicular,

(a) show that $q = 3$

(2)

Given further that l_1 and l_2 intersect at point X,

find

(b) the value of p,

(5)

(c) the coordinates of X.

(2)

The point A lies on l_1 and has position vector $\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$

Given that point B also lies on l_1 and that $AB = 2AX$

(d) find the two possible position vectors of B.

(3)

a) $\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} q \\ 2 \\ 1 \end{pmatrix} = -2q + 2 + 4 = 0$

$q = \frac{-6}{-2} = 3$

b) At X, $14 - 2\lambda = p + 3\mu$ (1)

$-6 + \lambda = -7 + 2\mu$ (2)

$-13 + 4\lambda = 4 + \mu$ (3)

$2\times(3) - (2)$: $-35 + 7\lambda = 0$

$\lambda = 5$

In (3): $\mu = 4(5) - 17 = 3$ In (1): $p = 14 - 2(5) + 3(3) = \cancel{13}\ 5$

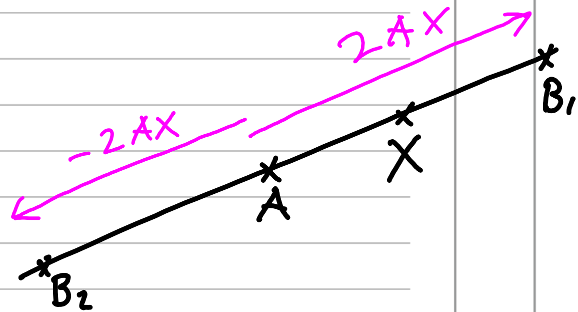
Leave blank

Question 11 continued

c) When $\lambda = 5$, $\underline{r} = \begin{pmatrix} 14-2(5) \\ -6+5 \\ -13+4(5) \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$

$\therefore X(4, -1, 7)$

d) $A(6, -2, 3)$, $AB = 2AX$



$\overrightarrow{AX} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$

$\underline{r}_{B_1} = \underline{r}_A + 2\overrightarrow{AX} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$

$\underline{r}_{B_2} = \underline{r}_A - 2\overrightarrow{AX} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ -5 \end{pmatrix}$

Leave blank

12.

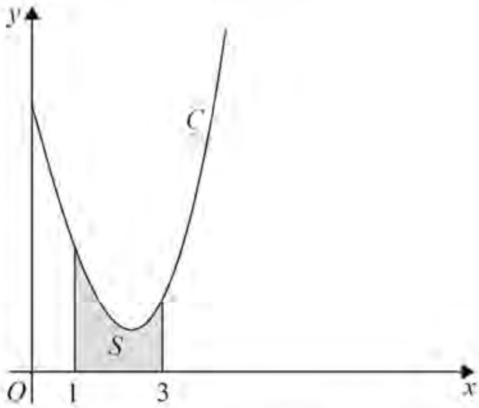


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 4, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the lines with equations $x = 1$ and $x = 3$

(a) Complete the table below with the value of y corresponding to $x = 2$. Give your answer to 4 decimal places.

x	1	1.5	2	2.5	3
y	2	1.3041	0.9242	0.9089	1.2958

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(c) Use calculus to find the exact area of S.

Give your answer in the form $\frac{a}{b} + \ln c$, where a, b and c are integers.

(6)

(d) Hence calculate the percentage error in using your answer to part (b) to estimate the area of S. Give your answer to one decimal place.

(2)

(e) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

Leave blank

Question 12 continued

a) When $x=2$, $y=\frac{(2)^2\ln 2}{3}-2(2)+4$

$=0.9242\ (4dp)$

b) $S=\frac{1}{2}(0.5)\left[2+1.2958+2(1.3041+0.9242+0.9089)\right]$

$=2.393$

c) $S=\int_1^3\left(\frac{x^2\ln x}{3}-2x+4\right)dx$ Let $u=\ln x$ $v'=x^2$

$u'=\frac{1}{x}$ $v=\frac{x^3}{3}$

$S=\frac{1}{3}\left[\frac{x^3\ln x}{3}\right]_1^3-\frac{1}{3}\int_1^3\frac{x^2}{3}dx+\left[-x^2+4x\right]_1^3$

$=\frac{3^3\ln 3}{9}-\frac{(1)\ln(1)}{9}-\left[\frac{x^3}{27}\right]_1^3-3^2+4(3)+(1)^2-4(1)$

$=3\ln 3-1+\frac{1}{27}$

$=\frac{-26}{27}+\ln 27$

d) $\text{Error}=\frac{\ln 27-\frac{26}{27}-2.393}{\ln 27-\frac{26}{27}}=2.6\%$

e) Use smaller intervals between x-values

Leave blank

13. (a) Express $10\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$

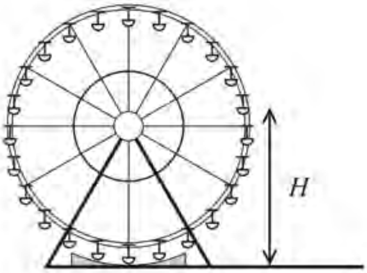
Give the exact value of R and give the value of α to 2 decimal places.

(3)

Alana models the height above the ground of a passenger on a Ferris wheel by the equation

$$H = 12 - 10\cos(30t)° + 3\sin(30t)°$$

where the height of the passenger above the ground is H metres at time t minutes after the wheel starts turning.



(b) Calculate

(i) the maximum value of H predicted by this model,

(ii) the value of t when this maximum first occurs.

Give each answer to 2 decimal places.

(4)

(c) Calculate the value of t when the passenger is 18 m above the ground for the first time.

Give your answer to 2 decimal places.

(4)

(d) Determine the time taken for the Ferris wheel to complete two revolutions.

(2)

a) $R\cos(\theta+\alpha) \equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\equiv 10\cos\theta - 3\sin\theta$

$\Rightarrow R\sin\alpha = 3$ $\qquad R = \sqrt{(-3)^2+10^2} = \sqrt{109}$

$R\cos\alpha = 10$

$\tan\alpha = \dfrac{3}{10}$

$\alpha = 16.70°$

$\therefore 10\cos\theta - 3\sin\theta \equiv \sqrt{109}\cos(\theta + 16.70)$

Leave blank

Question 13 continued

b) $H = 12 - 10\cos(30t) + 3\sin(30t)$

$= 12 - \sqrt{109}\,\underbrace{\cos(30t + 16.70)}_{\text{min. } -1}$

i) $H_{max} = 12 - \sqrt{109}(-1) = 22.44\text{m}$

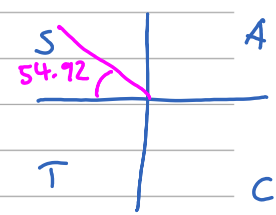
ii) For H_{max}, $\cos(30t + 16.70) = -1$

$30t + 16.70 = 180$

$t = 5.44\text{ min}$

c) $18 = 12 - \sqrt{109}\cos(30t + 16.70)$

$\cos(30t + 16.70) = \dfrac{-6}{\sqrt{109}}$



$30t + 16.70 = 125.08$

$t = 3.61\text{min}$

d) Period $= \dfrac{360}{30} = 12\text{min.}$

$\therefore$ It takes 24 mins to complete two revolutions