C3 June 2016 Model Answers
by Kprime2   TSR

1. The functions f and g are defined by

$$f : x \to 7x - 1, \qquad x \in \mathbb{R}$$

$$g : x \to \frac{4}{x - 2}, \qquad x \neq 2, x \in \mathbb{R}$$

(a) Solve the equation fg(x) = x

(4)

(b) Hence, or otherwise, find the largest value of a such that $g(a) = f^{-1}(a)$

(1)

1 (a). $fg(x) = 7\left(\frac{4}{x-2}\right) - 1$

$$= \frac{28}{x-2} - 1$$

$fg(x) = x \Rightarrow \quad \frac{28}{x-2} - 1 = x$

$\times x-2) \Rightarrow \quad 28 - x + 2 = x(x-2)$

$$\therefore \quad x^2 - x - 30 = 0$$

$$(x - 6)(x + 5) = 0$$

$$\Rightarrow \quad x = 6 \qquad x = -5$$

(b) $fg(a) = a$

$\Rightarrow f^{-1}[fg(a)] = f^{-1}(a)$ $\Big\}$ $a = 6$

$\therefore g(a) = f^{-1}(a)$ $\Big\}$

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2.
$$y = \frac{4x}{x^2 + 5}$$

(a) Find $\frac{dy}{dx}$, writing your answer as a single fraction in its simplest form.

(4)

(b) Hence find the set of values of x for which $\frac{dy}{dx} < 0$

(3)

2(a) $y = \dfrac{4x}{x^2 + 5}$

$$\frac{dy}{dx} = \frac{4(x^2+5) - 4x(2x)}{(x^2+5)^2}$$
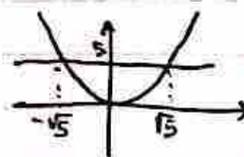
$$= \frac{4x^2 + 20 - 8x^2}{(x^2+5)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{20 - 4x^2}{(x^2+5)^2}$$

(b) $20 - 4x^2 < 0$

$\Rightarrow \quad x^2 > 5$

$x > \sqrt{5}$ $x < -\sqrt{5}$

3. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < 90°$. Give the exact value of R and give the value of α to 2 decimal places.

(3)

(b) Hence solve, for $0 \leqslant \theta < 360°$,

$$\frac{2}{2\cos\theta - \sin\theta - 1} = 15$$

Give your answers to one decimal place.

(5)

(c) Use your solutions to parts (a) and (b) to deduce the smallest positive value of θ for which

$$\frac{2}{2\cos\theta + \sin\theta - 1} = 15$$

Give your answer to one decimal place.

(2)

3(a) $R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\equiv 2\cos\theta - \sin\theta$

$\Rightarrow \quad R\cos\alpha = 2 \quad \& \quad R\sin\alpha = 1$

$R^2 = 2^2 + 1^2 = 5 \quad \Rightarrow \quad R = \pm\sqrt{5}$

$\tan\alpha = \frac{1}{2} \Rightarrow \alpha = \arctan\left(\frac{1}{2}\right) = 26.57° \,(2dp)$

$\therefore 2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta + 26.57°)$

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Question 3 continued

(b)
$$\frac{2}{2\cos\theta - \sin\theta - 1} = 15$$

$$\therefore \quad 2 = 30\cos\theta - 15\sin\theta - 15$$

$$17 = 15\left(2\cos\theta - \sin\theta\right)$$

$$\therefore \quad \sqrt{5}\cos(\theta + 26.57°) = \frac{17}{15}$$

$$0 \le \theta < 360° \qquad \therefore \cos(\theta + 26.57°) = \frac{17}{15\sqrt{5}}$$

$$26.57 \le (\theta + 26.57°) < 386.57°$$

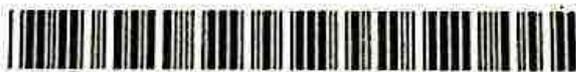$$\theta + 26.565\ldots = \arccos\left(\frac{17}{15\sqrt5}\right) = 59.54\ldots$$

~~$\theta + 26.565\ldots =$~~

$$< 180 - 59.54\ldots = 120.45\ldots$$

$$\theta + 26.565\ldots = 360 - 59.54\ldots = 300.453\ldots$$

$$\therefore \quad \underline{\theta = 33.0°} \qquad \underline{\theta = 273.9°}$$

Question 3 continued

(c) $2\cos\theta \pm \sin\theta = R\cos(\theta \mp \alpha)$

(a)

$\therefore 2\cos\theta + \sin\theta = \sqrt{5}\cos(\theta - 26.57)$

(b) from part (b):

$\cos(\theta - 26.57) = \dfrac{17}{15\sqrt{5}}$

$\therefore \theta - 26.565\ldots = \arccos\left(\dfrac{17}{15\sqrt{5}}\right)$

$= 59.546\ldots$

$\therefore \theta = 59.546\ldots + 26.565\ldots$
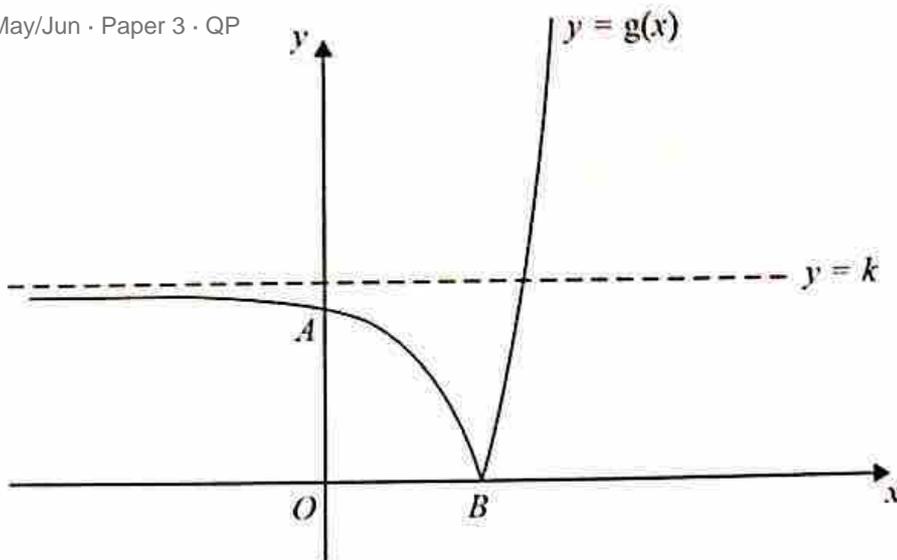
$\Rightarrow \theta = 86.1°\ (1dp)$

4.

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = g(x)$, where

$$g(x) = |4e^{2x} - 25|, \quad x \in \mathbb{R}$$

The curve cuts the y-axis at the point A and meets the x-axis at the point B. The curve has an asymptote $y = k$, where k is a constant, as shown in Figure 1

(a) Find, giving each answer in its simplest form,

　　(i) the y coordinate of the point A,

　　(ii) the exact x coordinate of the point B,

　　(iii) the value of the constant k.

(5)

The equation $g(x) = 2x + 43$ has a positive root at $x = \alpha$

(b) Show that α is a solution of $x = \dfrac{1}{2} \ln\left(\dfrac{1}{2}x + 17\right)$

(2)

The iteration formula

$$x_{n+1} = \frac{1}{2} \ln\left(\frac{1}{2}x_n + 17\right)$$

can be used to find an approximation for α

(c) Taking $x_0 = 1.4$ find the values of x_1 and x_2
Give each answer to 4 decimal places.

(2)

(d) By choosing a suitable interval, show that $\alpha = 1.437$ to 3 decimal places.

(2)

Question 4 continued

4 (a) (i) $g(0) = -21$

$$\therefore y_A = 21$$
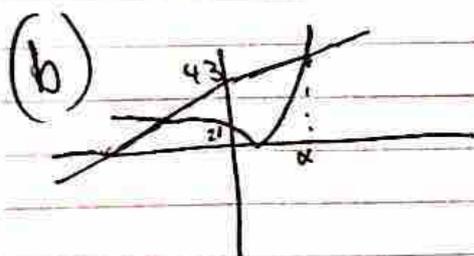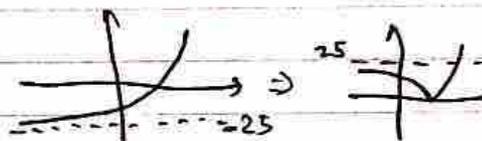
(ii) $g(x) = 0 \Rightarrow 4e^{2x} - 25 = 0$

$$e^{2x} = \frac{25}{4}$$

$$e^x = \frac{5}{2}$$

$$x = \ln\left(\frac{5}{2}\right)$$

(iii) $k = 25$

(b) $\Rightarrow 4e^{2x} - 25 = 2x + 43$

$$\therefore 4e^{2x} = 2x + 68$$

$$\therefore e^{2x} = \frac{2x+68}{4} = \frac{1}{2}x + 17$$

$$\therefore e^x = \sqrt{\frac{1}{2}x + 17}$$

$$\Rightarrow x = \ln\left(\sqrt{\frac{1}{2}x + 17}\right) = \ln\left[\left(\frac{1}{2}x + 17\right)^{\frac{1}{2}}\right]$$

$$\therefore x = \frac{1}{2}\ln\left(\frac{1}{2}x + 17\right) \quad \text{as required}$$

Question 4 continued

(C) $x_1 = 1.4368$ (4dp)

$x_2 = 1.4373$ (4dp)

(d) We are solving $4e^{2x} - 23 = 2x + 43$

∴ $4e^{2x} - 2x - 68 = 0$

Let $f(x) = 4e^{2x} - 2x - 68$

Solve $f(x) = 0$

$f(1.4365) = -0.11296\ldots$

$f(1.4375) = 0.026696\ldots$

There is a sign change in the interval $[1.4365, 1.4375]$

⇒ $\alpha \in [1.4365, 1.4375]$

∴ $\alpha \approx 1.437$ (3dp)

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5. (i) Find, using calculus, the x coordinate of the turning point of the curve with equation

$$y = e^{3x} \cos 4x, \quad \frac{\pi}{4} \leqslant x < \frac{\pi}{2}$$

Give your answer to 4 decimal places.

(5)

(ii) Given $x = \sin^2 2y, \quad 0 < y < \frac{\pi}{4}$, find $\dfrac{dy}{dx}$ as a function of y.

Write your answer in the form

$$\frac{dy}{dx} = p \operatorname{cosec}(qy), \qquad 0 < y < \frac{\pi}{4}$$

where p and q are constants to be determined.

(5)

5(i) $\quad y = e^{3x} \cos 4x \qquad \qquad \begin{aligned} u &= e^{3x} \\ v &= \cos 4x \end{aligned}$

$$\therefore \quad \frac{dy}{dx} = (\cos 4x)(3e^{3x}) + e^{3x}(-4\sin 4x)$$

$$= 3e^{3x} \cos 4x - 4e^{3x} \sin 4x$$

$$\frac{dy}{dx} = 0 \implies e^{3x}(3\cos 4x - 4\sin 4x) = 0$$

$$e^{3x} \neq 0$$

$$\therefore \quad 3\cos 4x - 4\sin 4x = 0$$

$$\therefore \quad 4\sin 4x = 3\cos 4x$$

$$\implies \quad \tan 4x = \frac{3}{4}$$

$\pi \leqslant 4x < 2\pi$ | $4x = \arctan\left(\frac{3}{4}\right) = 0.64$ ✗ reject

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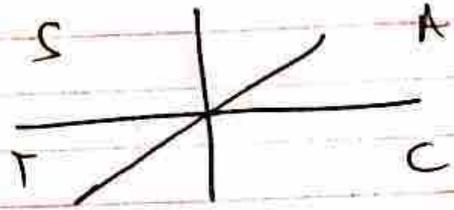**Question 5 continued**

$4x = 2\pi + 0.6435 = 6.9266\cdots$

$4x = \pi + \arctan\left(\frac{3}{4}\right)$

$\Rightarrow x = \frac{\pi}{4} + \frac{1}{4}\arctan\left(\frac{3}{4}\right)$

$\therefore x = 0.9463 \quad (4d\ell)$

(ii) $x = \sin^2 2y$

$\dfrac{\partial x}{\partial y} = 2\sin 2y \times 2\cos 2y = 4\sin 2y\cos 2y$

$\therefore \dfrac{\partial y}{\partial x} = \dfrac{1}{4\sin 2y\cos 2y}$

$\therefore \dfrac{\partial y}{\partial x} = \dfrac{1}{2(2\sin 2y\cos 2y)}$

$\therefore \dfrac{\partial y}{\partial x} = \dfrac{1}{2\sin 4y} = \frac{1}{2}\csc 4y$

$p = \frac{1}{2} \qquad q = 4$

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6. $$f(x) = \frac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6}, \qquad x > 2, x \in \mathbb{R}$$

(a) Given that

LHS $\quad \dfrac{x^4 + x^3 - 3x^2 + 7x - 6}{x^2 + x - 6} \equiv x^2 + A + \dfrac{B}{x - 2}$ RHS

find the values of the constants A and B.

(4)

(b) Hence or otherwise, using calculus, find an equation of the normal to the curve with equation $y = f(x)$ at the point where $x = 3$

(5)

(a)

$x > 2$:

$x = 3 \Rightarrow$ LHS $= 16$

$\qquad$ RHS $= 9 + A + B$

$\therefore \quad 9 + A + B = 16 \Rightarrow A + B = 7 \quad (1)$

$x = 4 \Rightarrow$ LHS $= 21$

$\qquad$ RHS $= 16 + A + \dfrac{B}{2}$

$\therefore \quad 16 + A + \dfrac{B}{2} = 21$

$\Rightarrow \quad 2A + B = 10 \quad (2)$
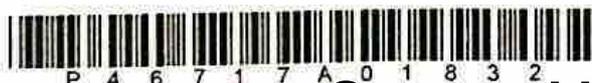
Solve simultaneously: $\quad A + A + B = 10$

$\therefore \quad A + 7 = 10$

$\Rightarrow A = 3$

$\quad B = 4$

$\underline{A = 3} \qquad \underline{B = 4}$

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Question 6 continued

(6) $f(x) = x^2 + 3 + \dfrac{4}{x-2}$

$f(3) = 16$

$f'(x) = 2x - 4(x-2)^{-2}$

$f'(3) = 2$

$\therefore$ gradient of normal $= -\dfrac{1}{2}$

$y - y_1 = m(x - x_1)$

$\therefore \quad y - 16 = -\dfrac{1}{2}(x - 3)$

$32 - 2y = x - 3$

$y = \dfrac{35 - x}{2}$

7. (a) For $-\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2}$, sketch the graph of $y = g(x)$ where
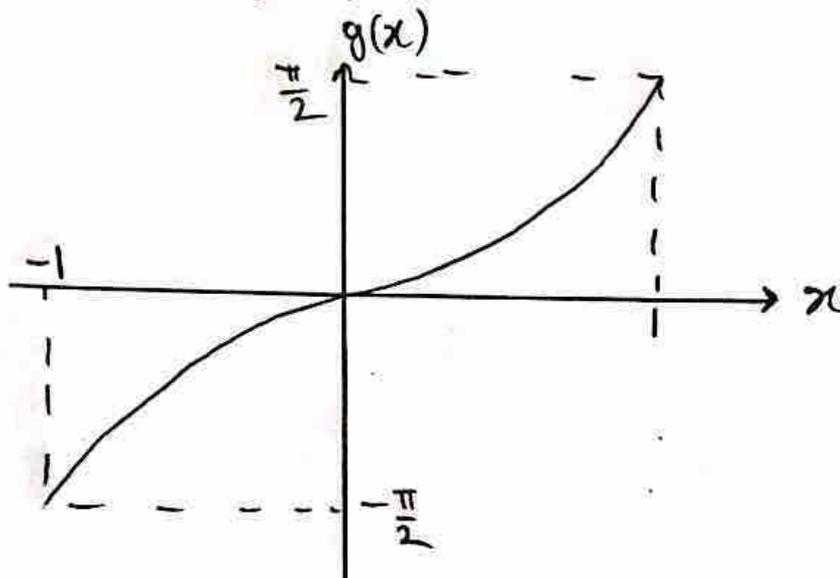
$$g(x) = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

(2)

(b) Find the exact value of x for which

$$3g(x + 1) + \pi = 0$$

(3)

(b) $\quad 3g(x+1) = -\pi$

$\qquad g(x+1) = -\dfrac{\pi}{3}$

$\qquad g^{-1}\left(g(x+1)\right) = g^{-1}\left(-\dfrac{\pi}{3}\right)$

$\therefore \quad x+1 = g^{-1}\left(-\dfrac{\pi}{3}\right)$

$\therefore \quad x+1 = \sin\left(-\dfrac{\pi}{3}\right)$

$\qquad x = \sin\left(-\dfrac{\pi}{3}\right) - 1$

$\qquad \Rightarrow x = \dfrac{-2-\sqrt{3}}{2}$

8. (a) Prove that

$$2\cot 2x + \tan x \equiv \cot x \qquad x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

(4)

(b) Hence, or otherwise, solve, for $-\pi \leqslant x < \pi$,

$$6\cot 2x + 3\tan x = \operatorname{cosec}^2 x - 2$$

Give your answers to 3 decimal places.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

8(a)

$$\text{LHS} = 2\cot 2x + \tan x = \frac{2\cos 2x}{\sin 2x} + \frac{\sin x}{\cos x}$$

$$= \frac{2}{\tan 2x} + \tan x$$

$$= \frac{2}{\frac{2\tan x}{1-\tan^2 x}} + \tan x$$

$$= \frac{1-\tan^2 x}{\tan x} + \tan x$$

$$= \frac{1-\tan^2 x}{\tan x} + \frac{\tan^2 x}{\tan x}$$

$$= \frac{1-\tan^2 x + \tan^2 x}{\tan x} = \frac{1}{\tan x} = \cot x$$

$$= \text{RHS} \quad \text{as required.}$$

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Question 8 continued

(b) $6\cot 2x + 3\tan x = \csc^2 x - 2$

$\therefore 3(2\cot 2x + \tan x) = \csc^2 x - 2$

$\therefore 3\cot x = \csc^2 x - 2$
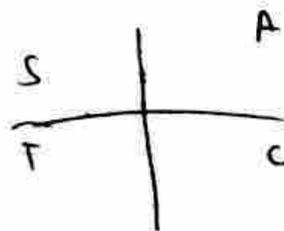
$s^2 + c^2 = 1$

$\dfrac{t^2 + 1}{1 + \cot^2} = \csc^2$

$3\cot x = \cot^2 x - 1$

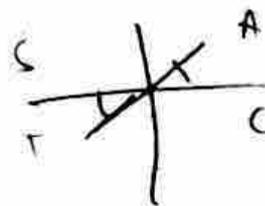$\therefore \cot^2 x - 3\cot x - 1 = 0$

$\cot x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore \cot x = \dfrac{3 \pm \sqrt{13}}{2}$

$\therefore \tan x = \dfrac{2}{3 \pm \sqrt{13}}$

$\tan x = \dfrac{2}{3 + \sqrt{13}}$

$x = \arctan\left(\dfrac{2}{3 + \sqrt{13}}\right) = 0.294$

$x = -\pi + 0.294001\ldots = -2.848$

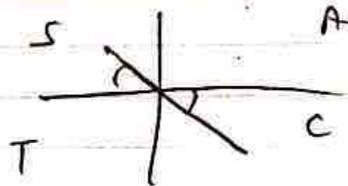**Question 8 continued**

$$\tan x = \frac{2}{3 - \sqrt{13}}$$

$$x = \arctan\left(\frac{2}{3 - \sqrt{13}}\right) = -1.277$$

$$x = \pi - 1.2767.7 \cdots = 1.865$$

$$\therefore \quad x = -2.848^c \qquad x = 0.294^c$$

$$x = -1.277^c \qquad x = 1.865^c$$

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9. The amount of an antibiotic in the bloodstream, from a given dose, is modelled by the formula

$$x = De^{-0.2t}$$

where x is the amount of the antibiotic in the bloodstream in milligrams, D is the dose given in milligrams and t is the time in hours after the antibiotic has been given.

A first dose of 15 mg of the antibiotic is given.

(a) Use the model to find the amount of the antibiotic in the bloodstream 4 hours after the dose is given. Give your answer in mg to 3 decimal places.

(2)

A second dose of 15 mg is given 5 hours after the first dose has been given. Using the same model for the second dose,

(b) show that the **total** amount of the antibiotic in the bloodstream 2 hours after the second dose is given is 13.754 mg to 3 decimal places.

(2)

No more doses of the antibiotic are given. At time T hours after the second dose is given, the total amount of the antibiotic in the bloodstream is 7.5 mg.

(c) Show that $T = a\ln\left(b + \dfrac{b}{c}\right)$, where a and b are integers to be determined.

(4)

9(a) $D = 15 \quad t = 4$

$$x = 15e^{-0.2 \times 4} = 6.740 \ (3dp)$$

(b) 7 hours since first dose:

$$x = 15e^{-0.2 \times 7} = 3.698\ldots$$

2 hours since second dose:

$$x = 15e^{-0.2 \times 2} = 10.054\ldots$$

$$\therefore \sum x = 15e^{-0.2 \times 7} + 15e^{-0.2 \times 2}$$

$$= 13.7537\ldots = 13.754 \ (3dp)$$

as required.

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DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

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Question 9 continued

$5 + T$ hours since first dose

T hours since second dose

$$\therefore \sum n = 15e^{-0.2(5+T)} + 15e^{-0.2T} = 7.5$$

$\left(X e^{0.2T} \right)$

$$\Rightarrow 15e^{-0.2T-1} e^{0.2T} + 15 = 7.5 e^{0.2T}$$

$$\therefore 15e^{-1} + 15 = 7.5 e^{0.2T}$$

$\left(\div 7.5 \right) \Rightarrow \dfrac{2}{e} + 2 = e^{0.2T}$

$$\ln(e^{0.2T}) = \ln\left(\dfrac{2}{e} + 2\right)$$

$$\therefore 0.2T = \ln\left(\dfrac{2}{e} + 2\right)$$

$$\Rightarrow T = 5\ln\left(2 + \dfrac{2}{e}\right)$$

$$\underline{a = 5} \quad \underline{b = 2}$$