C34 Jan 2017 (MA)

Q1) $\frac{d}{dx}\left[x^3 + 3x^2y + y^3 = 37\right]$

$\Rightarrow 3x^2 + 3x^2\frac{dy}{dx} + 6xy + 3y^2\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}(3x^2 + 3y^2) = -3x^2 - 6xy$

$\Rightarrow \frac{dy}{dx} = \frac{-3(x^2 + 2xy)}{3x^2 + 3y^2} = \frac{-(x^2 + 2xy)}{x^2 + y^2}$ //

at $(1, 3)$, $\frac{dy}{dx} = \frac{-(1 + 2(3))}{1 + 9} = -\frac{7}{10}$ //

$\Rightarrow y - 3 = -\frac{7}{10}(x - 1)$

$\Rightarrow y - 3 = -\frac{7}{10}x + \frac{7}{10}$

$\Rightarrow y = -\frac{7}{10}x + \frac{37}{10}$

$\overset{\times 10}{\Rightarrow} 10y = -7x + 37$

$\Rightarrow 7x + 10y - 37 = 0$

Q2a) $f(x) = x^3 - 5x + 16 = 0$

$x^3 = 5x - 16$

$x = (5x-16)^{\frac{1}{3}}$ // $a = 5$, $b = -16$

b) $x_1 = -3$

$x_2 = (5(-3)-16)^{\frac{1}{3}} \simeq -3.1414..$

$x_3 = (5(-3.1414)-16)^{\frac{1}{3}} \simeq -3.16508...$

$x_4 = (5(-3.16508)-16)^{\frac{1}{3}} \simeq -3.16902..$

so $\boxed{\begin{array}{l} x_1 = -3.141 \\ x_2 = -3.165 \\ x_3 = -3.169 \end{array}}$

c) $f(-3.165) = (-3.165)^3 - 5(-3.165) + 16 = 0.12048...$

$f(-3.175) = (-3.175)^3 - 5(-3.175) + 16 = -0.13098..$

[change in sign between $x = -3.165$ and $x = -3.175$ $\therefore$ a root α lies between these values hence $\alpha = -3.17$ to 2 d.p.]

Q3a) $\dfrac{9+11x}{(1-x)(3+2x)} = \dfrac{A}{1-x} + \dfrac{B}{3+2x}$

$9+11x = A(3+2x) + B(1-x)$

$\underline{x=1}$: $20 = 5A \therefore A = 4$ //

$\underline{x=0}$: $9 = 3A + B \therefore B = 9-12 = -3$ //

so $\dfrac{9+11x}{(1-x)(3+2x)} = \dfrac{4}{1-x} - \dfrac{3}{3+2x}$ //

① ②

b) ① : $4(1-x)^{-1} \simeq \left[1 + x + \dfrac{-1(-2)}{2}(-x)^2 + \dfrac{-1(-2)(-3)}{6}(-x)^3\right] \times 4$

$\begin{bmatrix} n=-1 \\ x=-x \end{bmatrix}$

$\simeq 4 \times [1 + x + x^2 + x^3]$

$\simeq 4 + 4x + 4x^2 + 4x^3$ //

② : $3(3+2x)^{-1} = 3 \cdot 3^{-1}(1+\frac{2}{3}x)^{-1} = (1+\frac{2}{3}x)^{-1}$ //

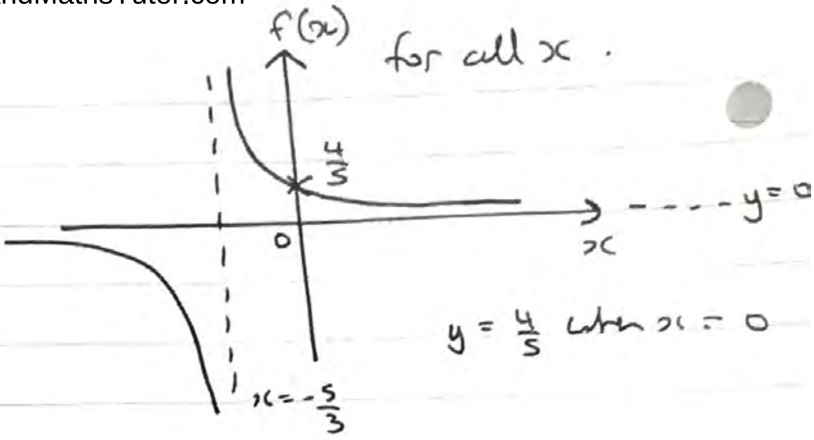
$\Rightarrow (1+\frac{2}{3}x)^{-1} \simeq 1 - \frac{2}{3}x + \dfrac{-1(-2)}{2}\left(\frac{2}{3}x\right)^2 + \dfrac{-1(-2)(-3)}{6}\left(\frac{2}{3}x\right)^3$

$\begin{bmatrix} x = \frac{2}{3}x \\ n=-1 \end{bmatrix}$

$\simeq 1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3$ //

$\therefore \dfrac{9+11x}{(1-x)(3+2x)} \simeq 4 + 4x + 4x^2 + 4x^3 - \left[1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3\right]$

$\simeq \boxed{3 + \frac{14}{3}x + \frac{32}{9}x^2 + \frac{116}{27}x^3}$



Q4a) $\boxed{0 < x < \frac{4}{5}}$

b) $y = \frac{4}{3x+5}$

$x \leftrightarrow y$; $x = \frac{4}{3y+5}$

make y subject : $(3y+5)x = 4$

$3xy + 5x = 4$

$y = \frac{4-5x}{3x}$ //

c) $g(x) = \frac{1}{x}$

$\therefore f[g(x)] = \frac{4}{3\left(\frac{1}{x}\right)+5} = \frac{4}{\frac{3}{x}+5} = \boxed{\frac{4x}{3+5x}}$

d) $fg(x) = \frac{4x}{3x+5}$

$gf(x) = \frac{1}{\frac{4}{3x+5}} = \frac{3x+5}{4}$ //

$\Rightarrow \frac{3x+5}{4} = \frac{4x}{3x+5}$

$\Rightarrow (3x+5)^2 = 16x$

$\Rightarrow 9x^2 + 30x + 25 - 16x = 0$

$\Rightarrow 9x^2 + 14x + 25 = 0$

$b^2 - 4ac = 14^2 - 4(9)(25) = -704 < 0$

$\therefore$ no real solutions exist.

Q5a)

x	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$
y	0	$\frac{7\pi\sqrt{2}}{8}$	2π	$\frac{9\pi\sqrt{2}}{8}$	0

$$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2} \rightarrow x\cos x = \frac{7\pi}{4}\cdot\frac{\sqrt{2}}{2} = \frac{7\pi\sqrt{2}}{8} //$$

$$\cos\frac{9\pi}{4} = \frac{\sqrt{2}}{2} \rightarrow x\cos x = \frac{9\pi}{4}\cdot\frac{\sqrt{2}}{2} = \frac{9\pi\sqrt{2}}{8} //$$

b) $h = \frac{b-a}{n} = \frac{\frac{5\pi}{2} - \frac{3\pi}{2}}{4} = \frac{\pi}{4} //$ (5 values so $n = 4$)

$$\text{Area} \simeq \frac{1}{2} \times \frac{\pi}{4}\left[0 + 0 + 2\left(\frac{7\pi\sqrt{2}}{8} + 2\pi + \frac{9\pi\sqrt{2}}{8}\right)\right]$$

$$\simeq \boxed{11.91} \text{ units}^2.$$

c) $\int(x\cos x)\,dx$ — By parts

$u = x \quad v' = \cos x$

$u' = 1 \quad v = \sin x$

$$\Rightarrow [x\sin x] - \int(\sin x)\,dx$$

$$\Rightarrow x\sin x + \cos x + c = \int(x\cos x)\,dx.$$

d) $$\left[x\sin x + \cos x\right]_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} = \left[\frac{5\pi}{2}(1) + (0)\right] - \left[-\frac{3\pi}{2} + 0\right]$$

$$= \frac{5\pi}{2} + \frac{3\pi}{2} = \boxed{4\pi}$$

Q6i) $y = 5x^2 \ln 3x$ PRODUCT RULE

$$\frac{dy}{dx} = 10x\ln 3x + 5x^2 \times \frac{1}{x}$$

$$= 10x\ln 3x + 5x = \boxed{5x(1+2\ln 3x)}$$

ii) $y = \dfrac{x}{\sin x + \cos x}$ QUOTIENT RULE

$u = x \quad u' = 1$

$v = (\sin x + \cos x) \quad v' = \cos x - \sin x$

$\Downarrow$

$$\frac{dy}{dx} = \frac{\sin x + \cos x - x(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x + x\sin x + \cos x - x\cos x}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x) + x(\sin x - \cos x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x(1+x) + \cos x(1-x)}{(\sin x + \cos x)^2}$$

$\square$

Q7ai) $y = |ax + b|$

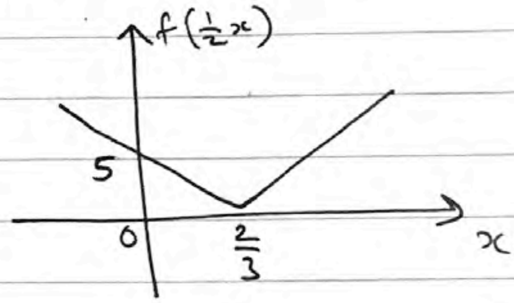
$(\frac{1}{3}, 0)$: $0 = |\frac{1}{3}a + b|$

$\Rightarrow a + 3b = 0$ //

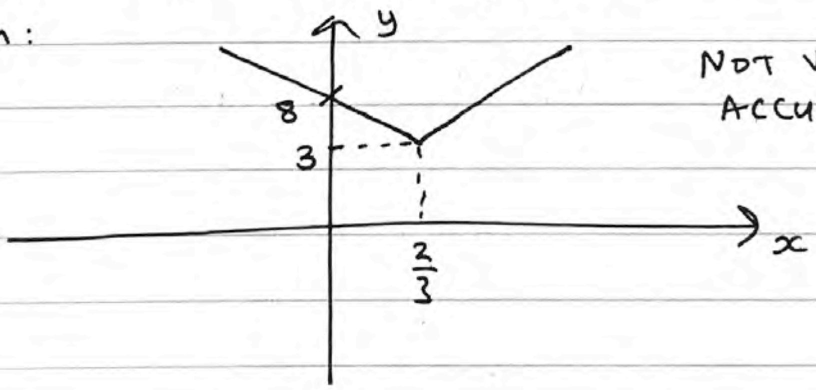
$(0, 5)$: $5 = |b|$ $\therefore b = 5 \downarrow a = -3(5) = -15$ // or $b = -5 \downarrow a = -3(-5) = 15$ //

ii) $y = |-15x + 5|$ or $y = |15x - 5|$

b) $f(\frac{1}{2}x)$ means multiply all x-values by 2.



$f(\frac{1}{2}x) + 3$ means multiply all x by 2 AND translation of 3 in the positive y-direction:



NOT VERY ACCURATE...

(Q8a) $\tan(2x + x) = \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$

$$= \frac{\left(\frac{2\tan x}{1-\tan^2 x}\right) + \tan x}{1 - \left(\frac{2\tan^3 x}{1-\tan^3 x}\right)} \quad \begin{matrix} \times (1-\tan^2 x) \\ \times (1-\tan^2 x) \end{matrix}$$

$$= \frac{2\tan x + \tan x(1-\tan^2 x)}{(1-\tan^2 x) - 2\tan^2 x} = \boxed{\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}}$$

b) $\text{LHS} = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = 11\tan x$

$\Rightarrow 11\tan x(1 - 3\tan^2 x) = 3\tan x - \tan^3 x$

$\Rightarrow 11\tan x - 33\tan^3 x + \tan^3 x - 3\tan x = 0$

$\Rightarrow \tan x(8 - 32\tan^2 x) = 0$ //

$\tan x = 0$

$x = \tan^{-1}(0) = \boxed{0^\circ} //$

(no other solutions)

$8 - 32\tan^2 x = 0$

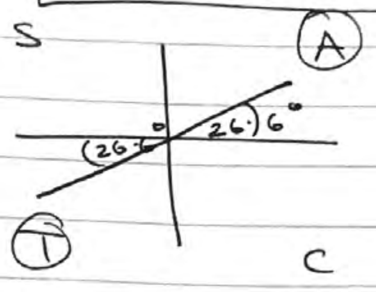
$\tan^2 x = \dfrac{8}{32}$

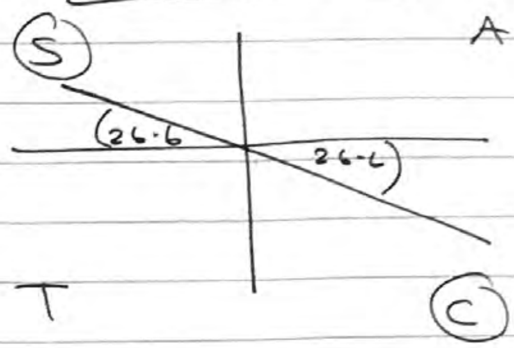
$\tan x = \pm\sqrt{\dfrac{8}{32}}$

$$\boxed{-30 < x < 30^\circ}$$

$\tan x = \sqrt{\frac{8}{32}}$ $\qquad$ $\tan x = -\sqrt{\frac{8}{32}}$

$\boxed{x = 26.6^\circ}$ $\qquad$ $\boxed{x = -26.6^\circ}$





Q9a) $\int_0^{12}\left[\frac{x}{(2x+3)^2}\right]dx$ $\qquad$ $u = 2x+3$

$\frac{du}{dx} = 2 \quad \therefore \quad dx = \frac{1}{2}du$ //

$u - 3 = 2x$

$\therefore x = \frac{1}{2}(u-3)$ //

x	u	
0	3	$2(0)+3=3$ //
12	27	$2(12)+3=27$ //

$$\Rightarrow \int_3^{27}\left[\frac{\frac{1}{2}(u-3)}{(u)^2}\times\frac{1}{2}\right]du$$

$$\Rightarrow \frac{1}{4}\int_3^{27}\left[\frac{u-3}{u^2}\right]du = \frac{1}{4}\int_3^{27}\left[\frac{1}{u}-\frac{3}{u^2}\right]du$$

$$\Rightarrow \frac{1}{4}\int_3^{27}\left[u^{-1}-3u^{-2}\right]du = \frac{1}{4}\left[\ln|u| \; \frac{-3u^{-1}}{-1}\right]_3^{27}$$

$$\Rightarrow \frac{1}{4}\left[\ln|u|+\frac{3}{u}\right]_3^{27} = \frac{1}{4}\left[\ln 27+\frac{3}{27}\right]-\frac{1}{4}\left[\ln 3+1\right]$$

$$\Rightarrow \frac{1}{4}\left[\ln 27-\ln 3\right]+\frac{1}{4}\left[\frac{1}{9}-1\right] = \frac{1}{4}\ln 9-\frac{2}{9}$$

$$\frac{1}{4}\ln 9 - \frac{2}{9} = \frac{1}{2}\ln(\sqrt{9}) - \frac{2}{9}$$

$$= \boxed{\frac{1}{2}\ln 3 - \frac{2}{9}}$$

b) $V = \pi\int_0^{12} y^2\,dx = 81\pi\int_0^{12}\frac{x}{(2x+3)^2}\,dx = 81\pi \times$ answer from (a)

$$\therefore V = 81\pi\left[\frac{1}{2}\ln 3 - \frac{2}{9}\right]$$

$$= \frac{81\pi}{2}\ln 3 - 18\pi \quad (o.e).$$

Q10a) $t=0$: $N = \dfrac{300}{3+17} = \dfrac{300}{20} = \boxed{15}$

b) $t=10$: $N = \dfrac{300}{3+17e^{-2}} = 56.5963\ldots$

$\therefore$ number of insects $= \boxed{56}$
(not quite at 57!)
so round down.

c) $82 = N$: $82 = \dfrac{300}{3+17e^{-0.2t}}$

$$\Rightarrow \frac{300}{82} = 3+17e^{-0.2t}$$

$$\Rightarrow \frac{\frac{300}{82}-3}{17} = e^{-0.2t} = \frac{27}{697}$$

$$\Rightarrow -0.2t = \ln\left(\frac{27}{697}\right)$$

$$\Rightarrow t = \frac{\ln\left(\frac{27}{697}\right)}{-0.2} = \boxed{16.3 \text{ weeks}}$$

d) $N = 300(3+17e^{-0.2t})^{-1}$

$\therefore$ PRODUCT RULE : $\frac{dN}{dt} = -300(3+17e^{-0.2t})^{-2} \times (-0.2 \times 17e^{-0.2t})$

$$= \frac{60e^{-0.2t}}{(3+17e^{-0.2t})^2} \times 17$$

$$= \frac{1020e^{-0.2t}}{(3+17e^{-0.2t})^2}$$

//

$t=5$: $\frac{dN}{dt} = \frac{1020e^{-1}}{(3+17e^{-1})^2} = 4.382...$

$= \boxed{4}$ insects per week

Q11a) $35\sin x - 12\cos x \equiv R\sin(x-\alpha) \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Comparing coefficients : $35 = R\cos\alpha$ — ①
$12 = R\sin\alpha$ — ②

$\frac{②}{①}$: $\frac{\cancel{R}\sin\alpha}{\cancel{R}\cos\alpha} = \tan\alpha = \frac{12}{35}$ //

$\therefore \alpha = \tan^{-1}\left(\frac{12}{35}\right) = \boxed{0.3303^c}$

Finding R : $R = \sqrt{35^2+12^2} = \boxed{37}$

$\therefore 35\sin x - 12\cos x \equiv \boxed{37\sin(x - 0.3303^c)}$

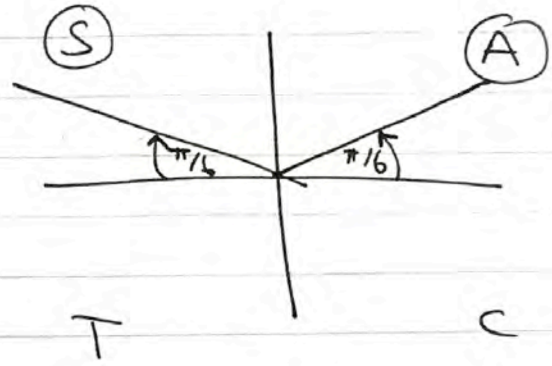
b) $70\sin x - 24\cos x = 2(35\sin x - 12\cos x)$

$\Rightarrow 74\sin(x - 0.3303^c) = 37$

$\div 74$
$\Rightarrow \sin(x - 0.3303^c) = \frac{1}{2}$

$\Rightarrow x - 0.3303^c = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ //

Solving in : $0 \leq x < 2\pi$



$x - 0.3303^c = \frac{\pi}{6}, \left(\pi - \frac{\pi}{6}\right)$

$x - 0.3303^c = \frac{\pi}{6}, \frac{5\pi}{6}$

$\boxed{x = 0.854^c, 2.95^c}$

c i) $y = \dfrac{7000}{31 + (37\sin(x - 0.3303^c))^2}$

Minimum value of y occurs when the denominator is the largest. (ie when $\sin(x - 0.3303^c) = 1$)

$\Rightarrow y_{min} = \dfrac{7000}{31 + (37)^2} = \boxed{5}$

ii) first maximum occurs at $\left(\frac{\pi}{2} = x\right)$ for $y = \sin x$.

$\Rightarrow x - 0.3303^c = \frac{\pi}{2}$

$\Rightarrow x = 0.3303 + \frac{\pi}{2} = \boxed{1.901^c}$

Q12a) $\left.\begin{matrix} t=0, & x=0 \\ t=2, & x=1.5 \end{matrix}\right\}$ $x = \frac{3}{4}t$ // $\longrightarrow$ $\boxed{t = \frac{4}{3}x}$

$\left(\frac{1.5}{2} = \frac{3}{4}\right)$

b) $\underline{x=3}$: $3 = \frac{3}{4}t \longrightarrow t = \frac{12}{3} = \boxed{4s}$

c) $\frac{dx}{dt} = \frac{\lambda}{2x+1}$

$(2x+1)\frac{dx}{dt} = \lambda$

$\int (2x+1)\,dx = \int (\lambda)\,dt$

$x^2 + x = \lambda t + c$

$\underline{t=0, x=0}$: $c = 0$ //

$\therefore \frac{x(x+1)}{\lambda} = t$ //

d) $\underline{x=1.5, t=2}$: $\frac{1.5(2.5)}{\lambda} = 2$

$\lambda = \frac{1.5 \times 2.5}{2} = \boxed{\frac{15}{8}}$

e) $t = \frac{3(3+1)}{\frac{15}{8}} = 6.4$ hours after 4pm //

$= \boxed{10:24 \text{ pm}}$

Q13a) $\underline{x=0}$: $1+\sqrt{3}\tan\theta = 0$

$$\tan\theta = \frac{-1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6} \quad //$$

$$y = \frac{5}{\cos\left(\frac{-\pi}{6}\right)} = \frac{10\sqrt{3}}{3} \quad // \quad \Big\} \boxed{\left(0, \frac{10\sqrt{3}}{3}\right)} \Leftarrow A$$

b) $\frac{dx}{d\theta} = \sqrt{3}\sec^2\theta$ $\qquad \frac{dy}{d\theta} = 5\sec\theta\tan\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cancel{\sec\theta}\tan\theta}{\sqrt{3}\sec^{\cancel{2}}\theta} = \frac{5\tan\theta}{\frac{\sqrt{3}}{\cos\theta}}$$

$$= \frac{\frac{5\sin\theta}{\cancel{\cos\theta}}}{\frac{\sqrt{3}}{\cancel{\cos\theta}}} = \frac{5}{\sqrt{3}}\sin\theta \quad //$$

$$\boxed{\therefore \lambda = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}}$$

c) $\underline{\frac{dy}{dx} = 0 \text{ at } B}$: $\frac{5}{\sqrt{3}}\sin\theta = 0$

$$\theta = 0 \quad //$$

$$\therefore x_B = 1+\sqrt{3}\tan(0) = 1$$
$$y_B = 5\sec(0) = 5 \quad \Big\} B(1, 5) \quad //$$

d) $x = 1+\sqrt{3}\tan\theta$

$$\frac{(x-1)}{\sqrt{3}} = \tan\theta \quad \therefore \frac{(x-1)^2}{3} = \tan^2\theta.$$

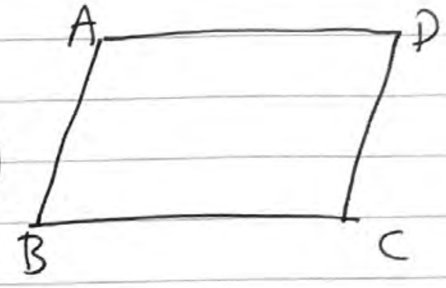
$$\Rightarrow \frac{(x-1)^2}{3} = \sec^2\theta - 1$$

$$\Rightarrow \frac{(x-1)^2+3}{3} = \sec^2\theta \quad //$$

d cont.) $\therefore \sec\theta = \sqrt{(x-1)^2+3} = \frac{1}{\sqrt{3}}\sqrt{x^2-2x+1+3}$

$\therefore 5\sec\theta = \boxed{\frac{5}{\sqrt{3}}\sqrt{x^2-2x+4} = y}$

Q14a) $\overrightarrow{BC} = \overrightarrow{AD}$



$\begin{pmatrix}-1\\3\\6\end{pmatrix} - \begin{pmatrix}3\\-1\\6\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}1\\1\\-2\end{pmatrix}$

$\overrightarrow{OC} - \overrightarrow{OB} = \overrightarrow{OD} - \overrightarrow{OA}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-4\\4\\0\end{pmatrix} + \begin{pmatrix}1\\1\\-2\end{pmatrix} = \boxed{\begin{pmatrix}-3\\5\\-2\end{pmatrix}} = \overrightarrow{OD} = d.$

b) $\overrightarrow{AB} = \begin{pmatrix}3\\-1\\6\end{pmatrix} - \begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}2\\-2\\8\end{pmatrix} \rightarrow |\overrightarrow{AB}| = \sqrt{2^2+2^2+8^2} = 6\sqrt{2}$ //

$\overrightarrow{CB} = \begin{pmatrix}3\\-1\\6\end{pmatrix} - \begin{pmatrix}-1\\3\\6\end{pmatrix} = \begin{pmatrix}4\\-4\\0\end{pmatrix} \rightarrow |\overrightarrow{CB}| = \sqrt{4^2+4^2} = 4\sqrt{2}$ //

Both vectors must either be going towards B or from B. using $\overrightarrow{BC}$ and $\overrightarrow{BA}$ would also be ok.

$\overrightarrow{AB} \cdot \overrightarrow{CB} = \begin{pmatrix}2\\-2\\8\end{pmatrix} \cdot \begin{pmatrix}4\\-4\\0\end{pmatrix} = 16$ //

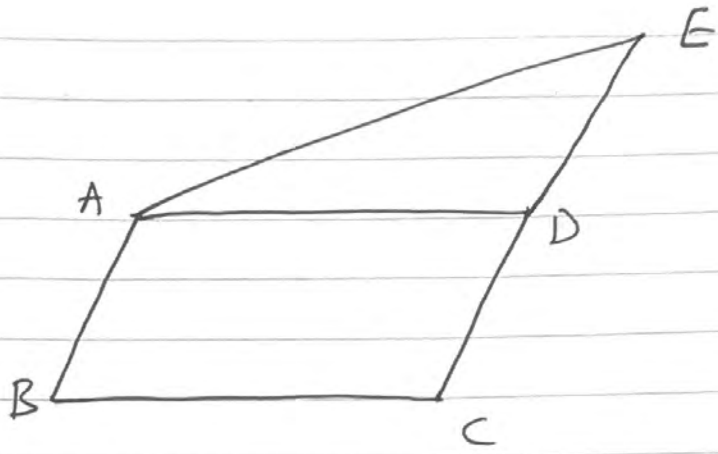
$\therefore \cos\theta = \frac{16}{6\sqrt{2} \times 4\sqrt{2}} = \frac{1}{3}$ //

$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right) = \boxed{70.5^\circ}$

c) Area $\Delta ABC = \frac{1}{2} \times 4\sqrt{2} \times 6\sqrt{2} \times \sin(70.5^\circ)$
$= 16\sqrt{2}$ //

$\therefore$ Area Parallelogram $= 2 \times 16\sqrt{2} = \boxed{32\sqrt{2}}$

d)



$\Delta ADE = \Delta ABC = \Delta ADC$

$\therefore$ Area $\Delta ADE = 16\sqrt{2}$ //

Total area $= 16\sqrt{2} + 32\sqrt{2} = \boxed{48\sqrt{2}}$