C34 October 2017 (MA)

Q1a) $f(x)=0$: $x^5 + x^3 - 12x^2 - 8 = 0$

$x^3 = 12x^2 + 8 - x^5$

$x^3 + x^5 = 4(3x^2+2)$

$x^3(1+x^2) = 4(3x^2+2)$

$\therefore\ x^3 = \dfrac{4(3x^2+2)}{x^2+1}$

hence $x = \sqrt[3]{\dfrac{4(3x^2+2)}{x^2+1}}$

b) $x_1 = \sqrt[3]{\dfrac{4(3(2)^2+2)}{(2)^2+1}} = \boxed{2.237}$

similarly, $x_2 = \boxed{2.246}$

$x_3 = \boxed{2.247}$

c) $\left.\begin{aligned} f(2.2465) &= -0.0057\ldots \\ f(2.2475) &= +0.083\ldots \end{aligned}\right\}$ change of sign between $x = 2.2465$ and $x = 2.2475$ //

hence $\alpha = 2.247$ to 3 dp

Q2a) $\frac{d}{dx}\left(y^3 + x^2y - 6x\right) = 0$

$\Rightarrow 3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy - 6 = 0$

$\Rightarrow \frac{dy}{dx}\left(x^2 + 3y^2\right) = 6 - 2xy$

$\Rightarrow \frac{dy}{dx} = \boxed{\frac{6 - 2xy}{x^2 + 3y^2}}$

b) $\underline{\frac{dy}{dx} = 0}$: $\frac{6 - 2xy}{x^2+3y^2} = 0$

$\therefore 6 - 2xy = 0$

$xy = 3$

$\therefore y = \frac{3}{x}$ //

Sub into eqn for C : $\left(\frac{3}{x}\right)^3 + x^2\left(\frac{3}{x}\right) - 6x = 0$

$\frac{27}{x^3} + 3x - 6x = 0$

$3x = \frac{27}{x^3}$

$\therefore x^4 = 9$ // $\therefore x = \pm\sqrt{3}$ //

if $x = \sqrt{3}$, $y = \frac{3}{\sqrt{3}} = \sqrt{3}$

if $x = -\sqrt{3}$, $y = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

$\therefore$ points are : $\boxed{(\sqrt{3}, \sqrt{3})}$ and $\boxed{(-\sqrt{3}, -\sqrt{3})}$

Q3a) $\underline{t=0}$: $N = 3500(1.035)^0 = \boxed{3500}$

b) $10000 = 3500(1.035)^t$

$\frac{10000}{3500} = \frac{20}{7} = 1.035^t$

$\ln\left[\frac{20}{7}\right] = \ln\left[(1.035)^t\right]$

$\therefore \quad t \ln(1.035) = \ln\left(\frac{20}{7}\right)$

$\Rightarrow \quad t = \frac{\ln\frac{20}{7}}{\ln 1.035} = 30.517$ hours

$= \boxed{30 \text{ hours } 31 \text{ min}}$

c) $\frac{dN}{dt} = 3500(1.035)^t \ln(1.035)$

$\underline{\text{at } t = 8}$: $\frac{dN}{dt} = 3500(1.035)^8(\ln 1.035)$

$= \boxed{159}$ 3 s.f.

Q4a) $\dfrac{1-\cos 2x}{\sin 2x} = \tan x$

LHS $= \dfrac{1-(1-2\sin^2 x)}{2\sin x\cos x} = \dfrac{2\sin^{\cancel{2}} x}{2\cancel{\sin x}\cos x} = \dfrac{\sin x}{\cos x}$ //

$= \tan x$

b) $3\sec^2\theta - 7 = \tan\theta$

$3(1+\tan^2\theta) - 7 = \tan\theta$

$1+\tan^2\theta = \sec^2\theta$

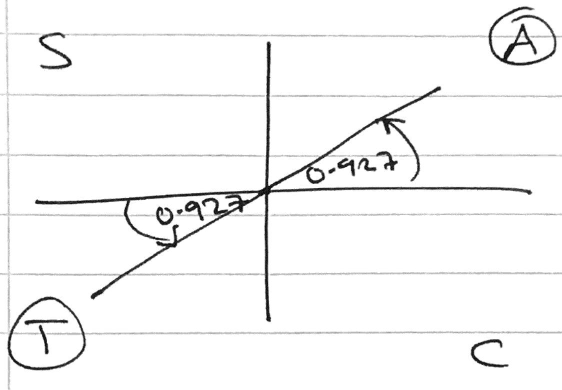
$\Rightarrow 3 + 3\tan^2\theta - 7 - \tan\theta = 0$

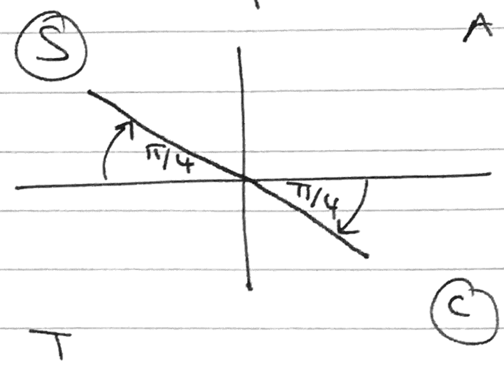
$\Rightarrow 3\tan^2\theta - \tan\theta - 4 = 0$

$\Rightarrow (3\tan\theta - 4)(\tan\theta + 1) = 0$

$3\tan\theta - 4 = 0$	$\tan\theta + 1 = 0$
$\tan\theta = \dfrac{4}{3}$ //	$\tan\theta = -1$ //
$\theta = \tan^{-1}\left(\dfrac{4}{3}\right) = \underline{0.927^c}$	$\theta = \tan^{-1}(-1) = \underline{-\dfrac{\pi}{4}}$

$\theta = 0.927^c$ $\theta = -\frac{\pi}{4}$





$\theta = 0.927, \pi + 0.927$
$\hookrightarrow 4.069$

$\theta = \frac{3\pi}{4}, 2\pi - \frac{\pi}{4}$

$$\boxed{\theta = 0.927^c, 4.069^c, 2.356^c, 5.498^c}$$

Q5i) $\int \left((3x+5)^9 + e^{5x} \right) dx$

$$= \frac{[3x+5]^{10}}{3 \times 10} + \frac{1}{5}e^{5x} + c$$

$$= \frac{1}{30}(3x+5)^{10} + \frac{e^{5x}}{5} + c$$

ii) $\int_2^b \left[\frac{x}{x^2+5} \right] dx = \left[\frac{1}{2} \ln(x^2+5) \right]_2^b$

By pattern!

$\frac{d}{dx}(x^2+5) = 2x = 2 \times$ numerator.

$$\therefore \left[\frac{1}{2}\ln[x^2+5]\right]_2^b = \ln\sqrt{6}$$

$$\frac{1}{2}\ln(b^2+5) - \frac{1}{2}\ln(9) = \ln\sqrt{6}$$

$$\frac{1}{2}\ln(b^2+5) - \ln 3 = \ln\sqrt{6}$$

$$\ln(b^2+5) = 2\ln 3 + 2\ln\sqrt{6}$$

$$\ln(b^2+5) = 2\ln(3\sqrt{6})$$

$$\therefore \ln(b^2+5) = \ln(54) \,//$$

$$\Rightarrow b^2+5 = 54$$

$$\Rightarrow b^2 = 49$$

$$\Rightarrow b = \pm 7 \,//$$

but $b > 2$ so $\boxed{b = 7}$

(Q6a)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	0.76679	a	0.15940	0

$$a = 2e^{-\frac{\pi}{2}}\sqrt{\sin\frac{\pi}{2}} = \boxed{0.41576}$$

b) $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$ //

$\therefore$ Area $\simeq \frac{1}{2} \times \frac{\pi}{4}\left[2(0.76679 + 0.41576 + 0.15940)\right]$

$\simeq \boxed{1.0540}$

c) $y = 2e^{-x}(\sin x)^{\frac{1}{2}}$ PRODUCT RULE

$\frac{dy}{dx} = -2e^{-x}(\sin x)^{\frac{1}{2}} + e^{-x}(\sin x)^{-\frac{1}{2}}(\cos x)$ //

$= \boxed{e^{-x}\left(\frac{\cos x}{\sqrt{\sin x}} - 2\sqrt{\sin x}\right)}$ // o.e

d) at turning point, $\frac{dy}{dx} = 0 \ldots$

$\Rightarrow e^{-x}\left(\frac{\cos x}{\sqrt{\sin x}} - 2\sqrt{\sin x}\right) = 0$

$\Rightarrow \frac{\cos x}{\sqrt{\sin x}} - 2\sqrt{\sin x} = 0$

$\Rightarrow \cos x - 2\sin x = 0$

$\div \cos x$

$\Rightarrow 1 - 2\tan x = 0$

$\Rightarrow \tan x = \frac{1}{2}$ // $\therefore x = \tan^{-1}\left(\frac{1}{2}\right)$

$= \boxed{0.464}$

Q7a) $(2-3x)^{-3} = 2^{-3}\left(1-\frac{3}{2}x\right)^{-3}$

$= \frac{1}{8}\left(1-\frac{3}{2}x\right)^{-3}$ // $\left[\begin{matrix} x = -\frac{3}{2}x \\ n = -3 \end{matrix}\right]$

$\frac{1}{8}\left(1-\frac{3}{2}x\right)^{-3} \simeq \frac{1}{8}\left[1 + \frac{9}{2}x + \frac{-3(-4)}{2}\left(-\frac{3}{2}x\right)^2\right]$

$\simeq \frac{1}{8}\left[1 + \frac{9}{2}x + \frac{27}{2}x^2\right]$

$\simeq \boxed{\frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2}$

b) $(4+ux)\left(\frac{1}{8} + \frac{9}{16}x + \frac{27}{16}x^2\right) = \frac{1}{2} + Ax + \frac{81}{16}x^2$

$\frac{1}{2} + \frac{9}{4}x + \frac{27}{4}x^2 + \frac{ux}{8} + \frac{9ux^2}{16} + \cancel{\frac{27ux^3}{16}}$

$= \frac{1}{2} + Ax + \frac{81}{16}x^2$

$\Rightarrow \frac{1}{2} + \frac{9}{4}x + \frac{27}{4}x^2 + \frac{ux}{8} + \frac{9ux^2}{16} = \frac{1}{2} + Ax + \frac{81}{16}x^2$

Comparing Coefficients

$\underline{x}$: $\frac{9}{4} + \frac{u}{8} = A$ ~ (1)

$\underline{x^2}$: $\frac{27}{4} + \frac{9u}{16} = \frac{81}{16}$

$\therefore u = \dfrac{\frac{81}{16} - \frac{27}{4}}{9/16} = -3$ //

c) so from ①, $A = \frac{9}{4} + \frac{(-3)}{8} = \boxed{\frac{15}{8}}$

(Q8) $\frac{2x^2-3}{x(x-1)} = A + \frac{B}{x} + \frac{C}{x-1}$ (partial fractions)

<u>finding A</u> :

$$\begin{array}{r|l} & 2 \\ x^2-x+0 & 2x^2+0x-3 \\ & \underline{2x^2-2x+0} \\ & 0+\underline{2x-3} \end{array}$$

$\leftarrow$ remainder

so $\frac{2x^2-3}{x(x-1)} = 2 + \frac{2x-3}{x(x-1)}$

$\frac{2x-3}{x(x-1)} = \frac{B}{x} + \frac{C}{x-1}$

$\therefore 2x-3 = B(x-1) + C(x)$

<u>let $x=0$</u> : $-3 = -B \quad \therefore B = 3$

<u>$x=1$</u> : $-1 = C$

so $\frac{2x-3}{x(x-1)} = \frac{-1}{x-1} + \frac{3}{x}$

so $\dfrac{2x^2-3}{x(x-1)} = \dfrac{3}{x} - \dfrac{1}{x-1} + 2$

$\therefore \displaystyle\int_3^4 \left[\frac{2x^2-3}{x(x-1)}\right] dx = \int_3^4 \left[2 + \frac{3}{x} - \frac{1}{x-1}\right] dx$

$= \left[2x + 3\ln|x| - \ln|x-1|\right]_3^4$

$= \left[8 + 3\ln 4 - \ln 3\right] - \left[6 + 3\ln 3 - \ln 2\right]$

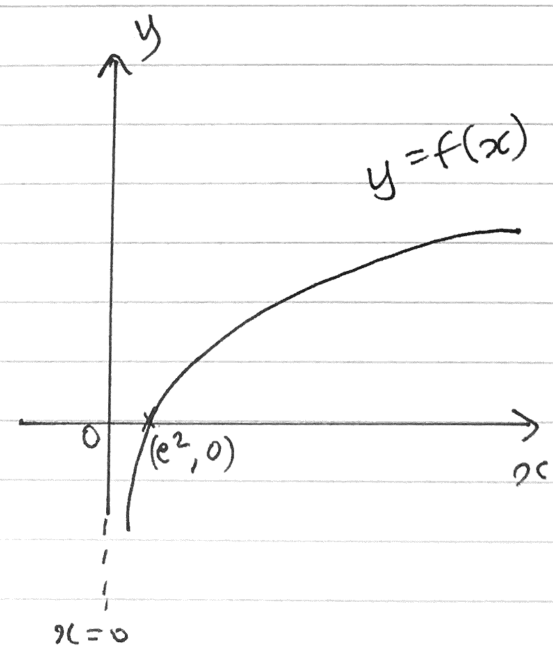
$= 2 + 3\ln 4 - 3\ln 3 + \ln 2 - \ln 3$

$= 2 + 3\ln 4 + \ln 2 - 4\ln 3$

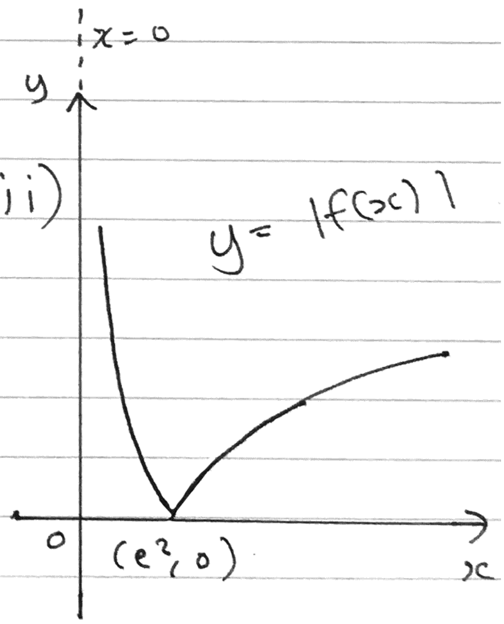
$= 2 + \ln(4^3) + \ln(2) - (\ln 3^4)$

$= \boxed{2 + \ln\frac{128}{81}}$

Q 9_{ai})



ii)



b) $|f(x)| = 4$

$2\ln(x) - 4 = 4$ and $-2\ln(x) + 4 = 4$

$\ln(x) = 4$ $\ln(x) = 0$

$\boxed{e^4 = x}$ $e^0 = \boxed{x = 1}$

c) $f(x) = 2\ln(x) - 4$

$g(x) = e^{x+5} - 2$

$gf(x) = g[f(x)] = e^{2\ln(x) - 4 + 5} - 2$

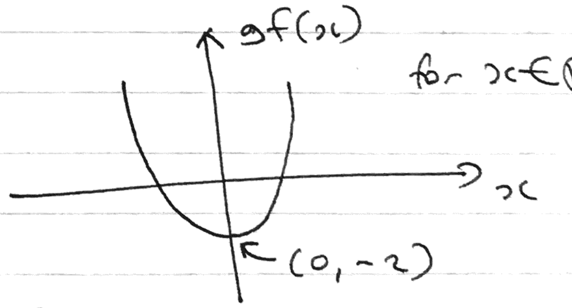
$= e^{2\ln(x) + 1} - 2$

$= e^{\ln(x^2) + 1} - 2$

$= x^2 \times e^1 - 2$

$= \boxed{ex^2 - 2}$

d) $\boxed{gf(x) > -2}$



for $x \in \mathbb{R}$.

note that the sign is not "$\geq$" as $f(x)$ is only defined for $x > 0$.

Q10a) $y=0$: $t(t-4)=0$

$t>0$, $\boxed{t=4}$ $\therefore x = \frac{20(4)}{2(4)-1} = \boxed{\frac{80}{9}}$ //

b) $x = 20t(2t+1)^{-1}$ PRODUCT RULE

$$\frac{dx}{dt} = 20(2t+1)^{-1} + 20t(-1)(2t+1)^{-2}(2)$$

$$\frac{dx}{dt} = \frac{20}{2t+1} - \frac{40t}{(2t+1)^2}$$

$$= \frac{20(2t+1) - 40t}{(2t+1)^2} = \frac{40t - 40t + 20}{(2t+1)^2}$$

$$= \frac{20}{(2t+1)^2} \quad //$$ and $y = t^2 - 4t$

$$\frac{dy}{dt} = 2t - 4 \quad //$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-4}{\frac{20}{(2t+1)^2}} = \frac{(2t-4)(2t+1)^2}{20} \quad //$$

$$= \frac{2(t-2)(2t+1)^2}{20} = \boxed{\frac{(t-2)(2t+1)^2}{10}}$$

c i) $x = \dfrac{20t}{2t+1}$

$2tx + x = 20t$

$t(2x - 20) + x = 0$

$t = \dfrac{-x}{2x-20} = \boxed{\dfrac{x}{20-2x} = t}$ //

ii) $y = t(t-4)$

$y = \left(\dfrac{x}{20-2x}\right)\left(\dfrac{x}{20-2x} - 4\right)$

$y = \left(\dfrac{x}{20-2x}\right)\left(\dfrac{x}{20-2x} - \dfrac{4(20-2x)}{20-2x}\right)$

$y = \dfrac{x}{20-2x}\left(\dfrac{x - 80 + 8x}{20-2x}\right)$

$y = \dfrac{x(9x-80)}{(20-2x)^2}$

denominater $\neq 0$. (anything $\div$ by 0 is undefined)

$x > 0 \quad \therefore \quad 20 - 2x > 0$

$20 - 2x > 0$

$x < 10$ // $\quad \therefore \boxed{k = 10}$

Q11a) $\int \left[\frac{1}{5-\sqrt{h}}\right] dh$

$u = 5 - h^{\frac{1}{2}}$

$\frac{du}{dh} = -\frac{1}{2} h^{-\frac{1}{2}}$

$dh = -2\sqrt{h}\, du.$

$\Rightarrow \int \left[\frac{1}{u} \times -2\sqrt{h}\right] du$

$\Rightarrow -2\int \left[\frac{\sqrt{h}}{u}\right] du = -2\int \left[\frac{5-u}{u}\right] du$

$\Rightarrow -2\int \left[\frac{5}{u} - 1\right] du = -2\left[5\ln|u| - [u]\right] + c$

$\Rightarrow -10\ln u + 2u + c$

$\Rightarrow 2(5-\sqrt{h}) - 10\ln(5-\sqrt{h}) + c$

$\Rightarrow 10 - 2\sqrt{h} + c - 10\ln(5-\sqrt{h})$

$10 + c$ is just a constant, we can call it k.

$\Rightarrow -2\sqrt{h} - 10\ln(5-\sqrt{h}) + k$ = result.

b) $$\frac{dh}{dt} = \frac{t^{0.2}(5-\sqrt{h})}{5}$$

$$\left(\frac{5}{5-\sqrt{h}}\right)\frac{dh}{dt} = t^{0.2}$$

$$5\int\left[\frac{1}{5-\sqrt{h}}\right]dh = \int(t^{0.2})\,dt.$$

$$5\left[-10\ln(5-\sqrt{h})-2\sqrt{h}\right] = \frac{t^{1.2}}{1.2}+k'$$

$\underline{t=0,\ h=2}$: $5\left[-10\ln(5-\sqrt{2})-2\sqrt{2}\right] = k'$

$\therefore\ k' = \left[-50\ln(5-\sqrt{2})-10\sqrt{2}\right]$

$$\therefore\ \left[-50\ln(5-\sqrt{h})-10\sqrt{h}\right] = \frac{t^{1.2}}{1.2}-50\ln(5-\sqrt{2})-10\sqrt{2}$$

$\underline{h=15}$: $\left[-50\ln(5-\sqrt{15})-10\sqrt{15}+50\ln(5-\sqrt{2})+10\sqrt{2}\right]\times1.2 = t^{1.2}$

$$\therefore\ t = \sqrt[1.2]{1.2\left(-50\ln(5-\sqrt{15})-10\sqrt{15}+50\ln(5-\sqrt{2})+10\sqrt{2}\right)}$$

$$t = \sqrt[1.2]{39.939\ldots} = \boxed{21.6}$$ years.

c) $h=15, \quad t = 21.6$

$\therefore \frac{dh}{dt} = \frac{(21.6)^{0.2}(5-\sqrt{15})}{5} = 0.42$

$= \boxed{42 \text{ cm per year}}$

Q12a) $\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}$ is part of the equations l_1 and l_2.

Hence they meet at $\boxed{\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}}$

b) <u>Using dir. vecters</u> : $\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 16 - 8 + 1$
$= 9$

$\left|\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}\right| = \sqrt{8^2+4^2+1^2} = 9$

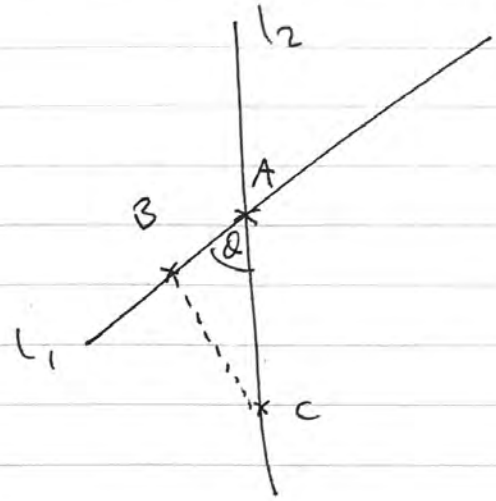
$\left|\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right| = \sqrt{2^2+2^2+1^2} = 3$

$\therefore \cos\theta = \frac{9}{3 \times 9} = \frac{1}{3}$

$\therefore \sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$

so $\sin\theta = \sqrt{\frac{8}{9}} = \boxed{\frac{2}{3}\sqrt{2}}$

c) $\vec{OB} = \begin{pmatrix} 2 + 2(4) \\ 0 - 2(4) \\ 7 + 1(4) \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 11 \end{pmatrix}$



$\vec{AB} = \vec{OB} - \vec{OA}$

$= \begin{pmatrix} 10 \\ -8 \\ 11 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -8 \\ 4 \end{pmatrix}$ //

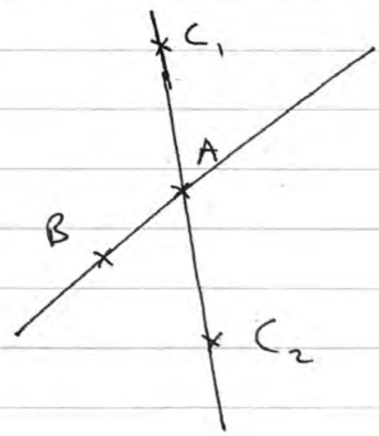
$\therefore |\vec{AB}| = \sqrt{8^2+8^2+4^2} = 12$ //

$\text{Area} = \frac{1}{2} ab\sin C = \frac{1}{2} \times AB \times AC \times \sin\theta$

$= \frac{1}{2} \times 12 \times 24 \times \sin\theta$

$= 144 \times \frac{2}{3}\sqrt{2} = \boxed{96\sqrt{2}}$

d) $\vec{AC} = \begin{pmatrix} 8\mu \\ 4\mu \\ \mu \end{pmatrix}$



$|\vec{AC}| = 24$ //

$\therefore \sqrt{(8\mu)^2 + (4\mu)^2 + (\mu)^2} = 24$

$\therefore \sqrt{81\mu^2} = 24 \quad \therefore 81\mu^2 = 576$

$\therefore \mu^2 = \frac{64}{9}$ //

hence $\mu = \sqrt{\frac{64}{9}} = \pm \frac{8}{3}$ //

so $\overrightarrow{OC} = \overrightarrow{OA} \pm \frac{8}{3}\begin{pmatrix} 8 \\ 4 \\ 1 \end{pmatrix}$

$$\overrightarrow{OC}_1 = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 64/3 \\ 32/3 \\ 8/3 \end{pmatrix} = \boxed{\begin{pmatrix} 70/3 \\ 32/3 \\ 29/3 \end{pmatrix}}$$

$$\overrightarrow{OC}_2 = \begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 64/3 \\ 32/3 \\ 8/3 \end{pmatrix} = \boxed{\begin{pmatrix} -58/3 \\ -32/3 \\ 13/3 \end{pmatrix}}$$