C34 January 2018 (MA)

Q1) $\frac{d}{dx}\left(3^x + xy = x + y^2\right)$

$$3^x \ln 3 + y + x\frac{dy}{dx} = 1 + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(x - 2y) = 1 - y - 3^x \ln 3$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 - y - 3^x \ln 3}{x - 2y}$$

At P(4, 11), $\quad \frac{dy}{dx} = \frac{1 - 11 - 3^4 \ln 3}{4 - 2(11)}$

$$= \frac{-10 - 81\ln 3}{-18}$$

$$= \boxed{\frac{5}{9} + \frac{9}{2}\ln 3}$$

Q2a) $(125-5x)^{\frac{2}{3}} = 125^{\frac{2}{3}}\left(1-\frac{5}{125}x\right)^{\frac{2}{3}}$

$= 25\left(1-\frac{x}{25}\right)^{\frac{2}{3}}$ $\quad \left[\begin{array}{l} x = -\frac{x}{25} \\ n = \frac{2}{3} \end{array}\right]$

$(125-5x)^{\frac{2}{3}} \simeq 25\left[1-\frac{2}{75}x+\frac{-\frac{2}{9}}{2}\left(\frac{-x}{25}\right)^2\right]$

$\simeq 25\left[1-\frac{2}{75}x-\frac{x^2}{5625}\right]$

$\simeq \boxed{25-\frac{2}{3}x-\frac{x^2}{225}}$

b) $125-5x = 120.$

$5x = 5$

$x = 1$ //

Sub $x=1$ into answer from (a):

$(120)^{\frac{2}{3}} \simeq 25-\frac{2}{3}(1)-\frac{(1)}{225}$

$\simeq \boxed{24.32889}$

Q3a) $\frac{x^2}{4} + \ln(2x) = 0$

$\ln(2x) = -\frac{x^2}{4}$

$2x = e^{-\frac{x^2}{4}}$

$\therefore x = \boxed{\frac{1}{2} e^{-\frac{x^2}{4}}}$

b) $x_2 = \frac{1}{2} e^{-\frac{1}{4}(0.5)^2} = \boxed{0.4697}$

similarly, $\boxed{x_3 = 0.4732}$

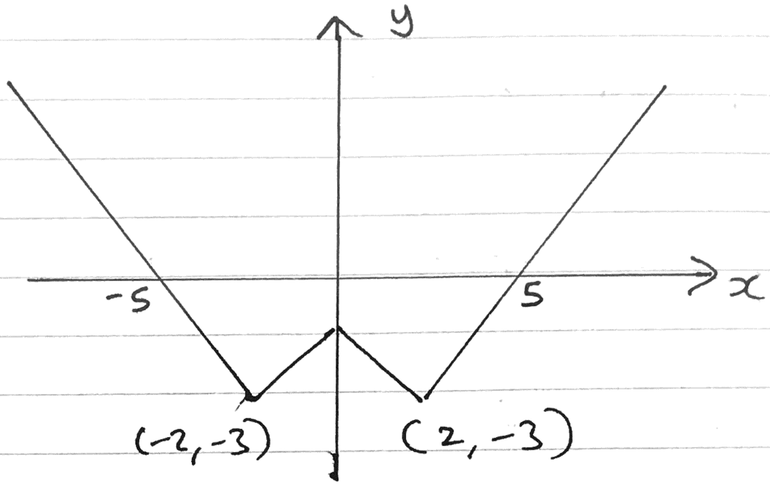
$\boxed{x_4 = 0.4728}$

c) $f(0.4725) = -0.000756\ldots$

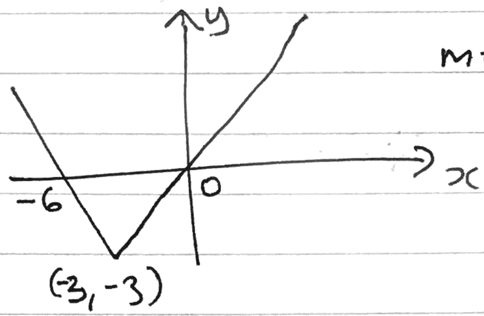
$f(0.4735) = 0.001594\ldots$

change in sign between $x = 0.4725$ and $x = 0.4735$ $\therefore$ a root of $f(x)$ is 0.473 to 3dp

Q4a)

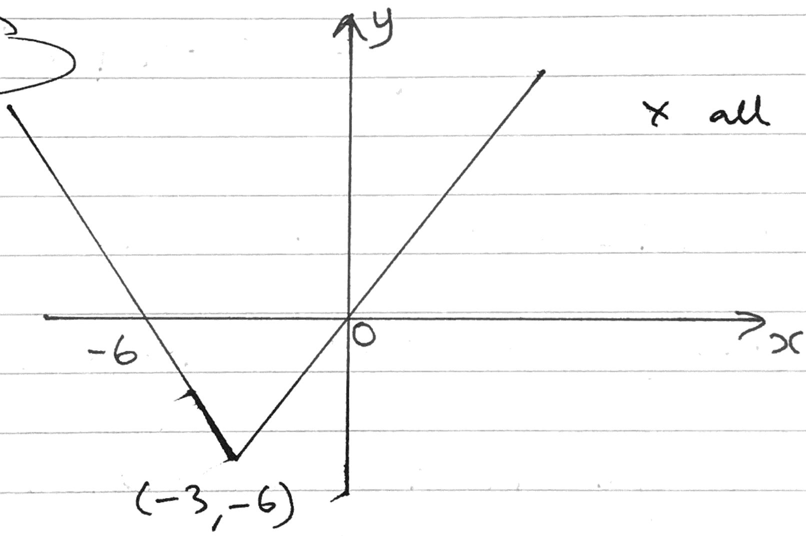


b) f(x+5) :



move everything 5 units to the left

2f(x+5)



x all y-values by 2

Q5a) $\dfrac{9(4+x)}{(4-3x)(4+3x)} = \dfrac{A}{4-3x} + \dfrac{B}{4+3x}$

$\therefore \; 9(4+x) = A(4+3x) + B(4-3x)$

$\underline{x = \frac{4}{3}}$: $48 = 8A \quad \therefore A = 6$

$\underline{x = 0}$: $36 = 4(6) + 4B$

$\therefore B = 3$

so $\dfrac{9(4+x)}{(4-3x)(4+3x)} = \dfrac{6}{4-3x} + \dfrac{3}{4+3x}$

b) $\displaystyle\int f(x)\,dx = 3\int\left[\frac{2}{4-3x} + \frac{1}{4+3x}\right]dx$

$= 3\left[-\frac{2}{3}\ln|4-3x| + \frac{1}{3}\ln|4+3x|\right] + c$

$= -2\ln|4-3x| + \ln|4+3x| + c$

$= \ln\left|\dfrac{4+3x}{(4-3x)^2}\right| + \ln e^c$

a constant

$$= \ln\left(\frac{A(4+3x)}{(4-3x)^2}\right)$$ where A is an arbitrary constant.

Q6) Volume required $= \pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(3\tan\left(\frac{x}{2}\right)\right) dx$

$$= 3\pi \int_{\pi/3}^{\pi/2} \left[\tan\left(\frac{x}{2}\right)\right] dx.$$

$$= 3\pi \left[2\ln\left(\sec\frac{x}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= 3\pi\left[2\ln\left|\sec\frac{\pi}{4}\right|\right] - 3\pi\left[2\ln\left|\sec\frac{\pi}{6}\right|\right]$$

$$= 6\pi\left[\ln\sqrt{2} - \ln\frac{2\sqrt{3}}{3}\right]$$

$$= 6\pi \times \ln\frac{\sqrt{6}}{2} = 3\pi \times \ln\left(\frac{\sqrt{6}}{2}\right)^2$$

$$= \boxed{3\pi \ln\frac{3}{2}}$$

$(A = 3\pi)$.

(Q7a) assuming l_1 and l_2 meet

$$\begin{pmatrix} 13+3\lambda \\ 15+3\lambda \\ -8-4\lambda \end{pmatrix} = \begin{pmatrix} 7+2\mu \\ -6-3\mu \\ 14+2\mu \end{pmatrix} \begin{matrix} \text{— ①} \\ \text{— ②} \\ \text{— ③} \end{matrix}$$

③ : $-8-4\lambda = 14+2\mu$

$\div 2$: $\mu + 7 = -4 - 2\lambda$

$\mu = -11 - 2\lambda$

→ ② : $15+3\lambda = -6-3(-11-2\lambda)$

$3\lambda = 12 + 6\lambda$

$\lambda = -4$

sub $\lambda = -4$ into ① : $13+3(-4) = 7+2\mu$

$\mu = \frac{-6}{2} = -3$

sub $\lambda = -4$ into ③ : $\mu = -11-2(-4) = -3$

values of μ and λ are consistent so l_1 and l_2 do indeed intersect

$$B = \begin{pmatrix} 7+2(-3) \\ -6-3(-3) \\ 14+2(-3) \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}}$$

b) using direction vectors :

$$\begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} = 6 - 9 - 8 = -11$$

$$\left\| \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \right\| = \sqrt{3^2+3^2+4^2} = \sqrt{34}$$

$$\left\| \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right\| = \sqrt{2^2+3^2+2^2} = \sqrt{17}$$

$$\therefore \cos\theta = \frac{-11}{\sqrt{34} \times \sqrt{17}} = \frac{-11}{17\sqrt{2}}$$

$$\cos^{-1}\left(\frac{-11}{17\sqrt{2}}\right) = 117.23°...$$

but we want the acute angle

$$180 - 117.23.. = \boxed{62.8°}$$

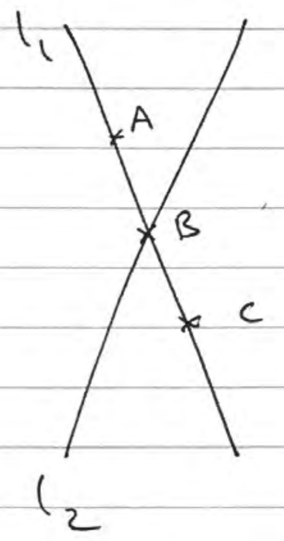
c) assume A lies on l_1 ...

$$\begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix} = \begin{pmatrix} 13 + 3\lambda \\ 15 + 3\lambda \\ -8 - 4\lambda \end{pmatrix} \begin{matrix} — ① \\ — ② \\ — ③ \end{matrix}$$

$$\left.\begin{array}{lll} ① : & 3\lambda = -18 & \therefore \lambda = -6 \\ ② : & 3\lambda = -18 & \therefore \lambda = -6 \\ ③ : & -4\lambda = 24 & \therefore \lambda = -6 \end{array}\right\}$$

values of λ are consistent so A does lie on l.

d) $\overrightarrow{OC} = \overrightarrow{OA} + 2\overrightarrow{AB}$



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 6 \\ 6 \\ -8 \end{pmatrix}$

$\therefore \overrightarrow{OC} = \begin{pmatrix} -5 \\ -3 \\ 16 \end{pmatrix} + 2\begin{pmatrix} 6 \\ 6 \\ -8 \end{pmatrix}$

$= \begin{pmatrix} -5+12 \\ -3+12 \\ 16-16 \end{pmatrix} = \boxed{\begin{pmatrix} 7 \\ 9 \\ 0 \end{pmatrix}}$

Q8) $y = 8\tan(2x)$

$$\frac{dy}{dx} = 16\sec^2 2x$$

$$\therefore \frac{dx}{dy} = \frac{1}{16\sec^2 2x}$$

$$\frac{y}{8} = \tan 2x$$

$$\left(\frac{y}{8}\right)^2 = \tan^2 2x$$

$$1 + \left(\frac{y}{8}\right)^2 = 1 + \tan^2 2x$$

$$1 + \tan^2 2x = \sec^2 2x$$

$$\therefore 1 + \left(\frac{y}{8}\right)^2 = 1 + \frac{y^2}{64} = \frac{64 + y^2}{64} = \sec^2 2x$$

$$\therefore \frac{dx}{dy} = \frac{1}{16\left(\frac{64+y^2}{64}\right)} = \frac{1}{\frac{64+y^2}{4}}$$

$$= \boxed{\frac{4}{64 + y^2}}$$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

Q9a) $\text{LHS} = \dfrac{\cot^2 x}{1+\cot^2 x} = \dfrac{\operatorname{cosec}^2 x - 1}{\operatorname{cosec}^2 x}$

$= \dfrac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} - \dfrac{1}{\operatorname{cosec}^2 x} \leftarrow \sin^2 x.$

$= 1 - \sin^2 x$

$= \cos^2 x$ as required.

b) $\cos^2 x = 8\cos 2x + 2\cos x$

$\cos^2 x = 8(2\cos^2 x - 1) + 2\cos x$

$16\cos^2 x - 8 + 2\cos x - \cos^2 x = 0$

$16\cos^2 x - \cos^2 x + 2\cos x - 8 = 0$

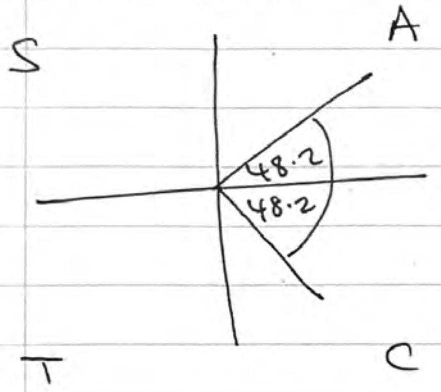
$15\cos^2 x + 2\cos x - 8 = 0$

By Quadratic Formula:

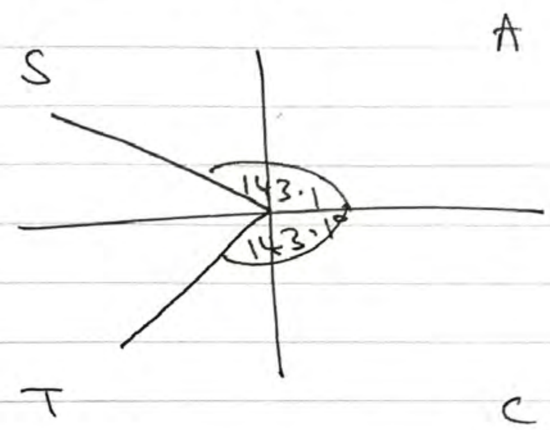
$\cos x = \dfrac{2}{3}$,	$\cos x = -\dfrac{4}{5}$
$x = \cos^{-1}\left(\dfrac{2}{3}\right) = 48.2^\circ$	$x = \cos^{-1}\left(-\dfrac{4}{5}\right) = 143.1^\circ$

$0 \le x < 360°$

$x = 48.2°$



$x = 143.1°$

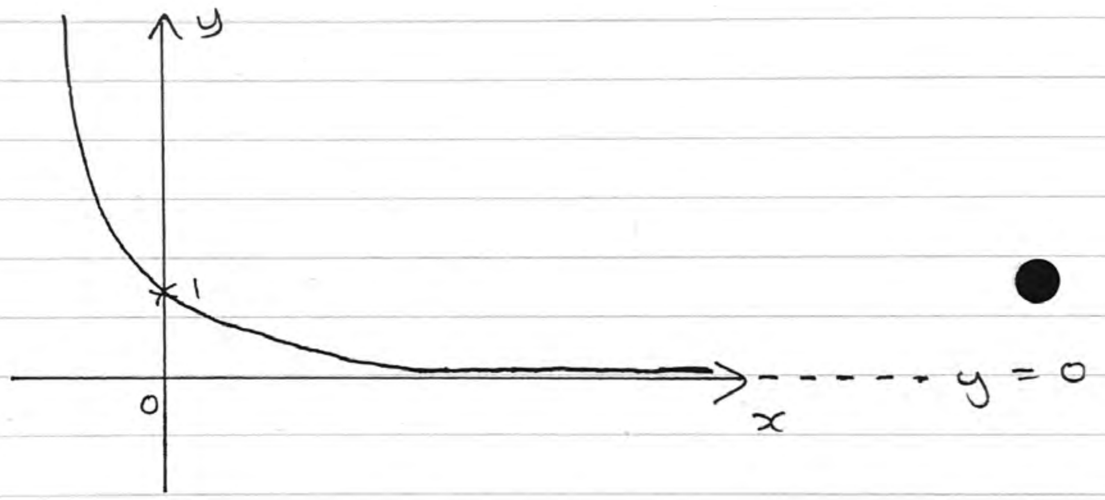


$x = 48.2°, 360 - 48.2°$

$x = 143.1°, 360 - 143.1$

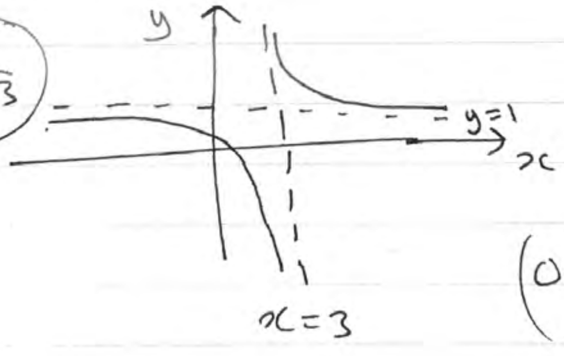
$$\boxed{x = 48.2°, 143.1°, 216.9°, 311.8°}$$

Q10a)



b) $\frac{x}{x-3} = \frac{x-3+3}{x-3} = 1 + \frac{3}{x-3}$

$y = 1 + \frac{3}{x-3}$



so range is $\boxed{y > 1}$

(or by observation of the equation)

c) $y = \dfrac{x}{x-3}$

$x \Longleftrightarrow y$; $x = \dfrac{y}{y-3}$

now make y the subject again:

$xy - 3x = y$

$y(x-1) = 3x$

$\therefore\ y = \dfrac{3x}{x-1} = g^{-1}(x)$ //

domain of $g^{-1}(x)$ = range of $g(x)$

$\therefore$ $\boxed{x > 1}$ is the domain.

d) $fg(x) = f[g(x)] = e^{-2\left(\frac{x}{x-3}\right)}$

$= e^{\frac{-2x}{x-3}}$ //

$\underline{fg(x) = 3}$: $e^{\frac{-2x}{x-3}} = 3$

$\Rightarrow \ln(3) = \ln\left(e^{\frac{-2x}{x-3}}\right)$

$\Rightarrow \dfrac{-2x}{x-3} = \ln 3$

$\Rightarrow x\ln 3 - 3\ln 3 = -2x$

$\Rightarrow x(\ln 3 + 2) = 3\ln 3$

$\Rightarrow \boxed{x = \dfrac{3\ln 3}{2+\ln 3}}$

at A x is minimum
at B x is max.

Q11a) at A / B, x is a max / min, $x = 3\cos t$

hence $x_{max} = 3$, $x_{min} = -3$ } as $-1 \leq \cos t \leq 1$

so $\boxed{A(-3, 0)}$ and $\boxed{B(3, 0)}$

b) $x = 0$ at $t = \frac{\pi}{2}, \frac{3\pi}{2}$

$y = 0$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

only values of t where both $x=0$ and $y=0$ are $\boxed{\frac{\pi}{2}}$ and $\boxed{\frac{3\pi}{2}}$

c) $x = 3\cos t \rightarrow \frac{dx}{dt} = -3\sin t$

$y = 9\sin 2t \rightarrow \frac{dy}{dt} = 18\cos 2t$

$\frac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\therefore \frac{dy}{dx} = \dfrac{18\cos 2t}{-3\sin t}$

$= \dfrac{18\cos\frac{\pi}{3}}{-3\sin\frac{\pi}{6}} = \boxed{-6}$

d) $y = 18\sin t\cos t$

$x = 3\cos t$

$\therefore \quad y = \frac{18}{3}\sin t \times x$ //

$x = 3\cos t$

$\frac{x}{3} = \cos t$

$\left(\frac{x}{3}\right)^2 = \cos^2 t$

$1 - \left(\frac{x}{3}\right)^2 = 1 - \cos^2 t = \sin^2 t$

$\therefore \quad 1 - \frac{x^2}{9} = \sin^2 t = \frac{9 - x^2}{9}$

$\therefore \quad \sin t = \frac{\sqrt{9 - x^2}}{3}$ //

so $\quad y = 6x \times \frac{\sqrt{9 - x^2}}{3} = 2x\sqrt{9 - x^2}$

$y^2 = \left(2x\sqrt{9 - x^2}\right)^2$

$\Rightarrow \boxed{y^2 = 4x^2(9 - x^2)}$

Q12a) $2\sin x - 4\cos x \equiv R\sin(x-\alpha) \equiv R\sin x\cos\alpha - R\cos x\sin\alpha$

Comparing coefficients: $2 = R\cos\alpha \sim (1)$

$4 = R\sin\alpha \sim (2)$

$\frac{(2)}{(1)}$: $\frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{4}{2} = 2$

$\therefore \alpha = \tan^{-1}(2) = \boxed{1.11^c}$

Finding R: $R = \sqrt{2^2+4^2} = \boxed{2\sqrt{5}}$

$\therefore 2\sin x - 4\cos x \equiv \boxed{2\sqrt{5}\sin(x - 1.11)}$

b) $H = 12 + 2\left(2\sqrt{5}\sin\left(\frac{2\pi t}{365} - 1.11\right)\right)$

H_{max} occurs when $\sin\left(\frac{2\pi t}{365} - 1.11\right) = 1$

$\therefore H_{max} = 12 + 4\sqrt{5}(1) = \boxed{20.9}$ hours

H_{min} occurs when $\sin\left(\frac{2\pi t}{365} - 1.11\right) = -1$

$\therefore H_{min} = 12 - 4\sqrt{5} = \boxed{3.06}$ hours

c) $17 = 12 + 4\sqrt{5} \sin\left(\frac{2\pi t}{365} - 1.11\right)$

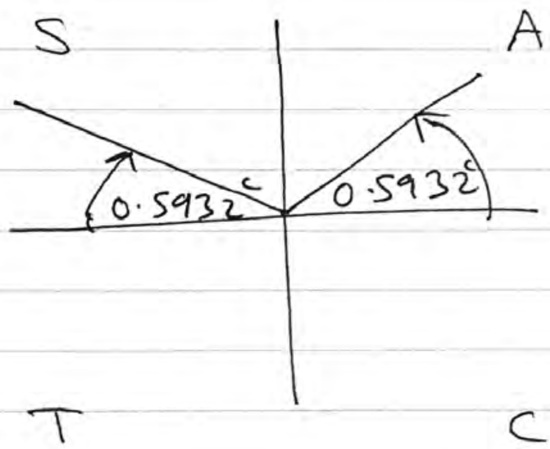
$\frac{5}{4\sqrt{5}} = \frac{\sqrt{5}}{4} = \sin\left(\frac{2\pi t}{365} - 1.11\right)$

$\frac{2\pi t}{365} - 1.11 = \sin^{-1}\left(\frac{\sqrt{5}}{4}\right) = 0.5932$

$0 \leq \frac{2\pi t}{365} \leq 2\pi$

$\Downarrow$

$\left(-1.11 \leq \frac{2\pi t}{365} - 1.11 \leq 2\pi - 1.11\right)$



$\therefore \frac{2\pi t}{365} - 1.11 = 0.593, 2.548$

$\frac{2\pi t}{365} = 1.703, 3.658$

$t = 98.9..., 212.5...$

$\boxed{\text{So } t = 99 \text{ days and } 213 \text{ days}}$

Q13a) $h = \frac{b-a}{n} = \frac{5e - e}{4} = e$

$\therefore \text{Area} \simeq \frac{1}{2} \times e \left[0.3114 + 0.1215 + 2(0.2195 + 0.1712 + 0.1416)\right]$

$\simeq \boxed{2.04}$ to 3sf.

b) $\int \frac{1}{2x} \ln 2x \, dx$

$\left[\begin{array}{l} u = \ln 2x \\ \frac{du}{dx} = \frac{2}{2x} = \frac{1}{x} \\ \therefore dx = x \, du. \end{array} \right]$

$\Rightarrow \int \left[\frac{1}{2x} \times u \times x\right] du = \int \left[\frac{1}{2}u\right] du$

$= \frac{u^2}{4} + c = \boxed{\frac{(\ln 2x)^2}{4} + c}$

c) $R = \left[\frac{(\ln 2x)^2}{4}\right]_e^{5e} = \left[\frac{(\ln 10e)^2}{4}\right] - \left[\frac{(\ln 2e)^2}{4}\right]$

$= \frac{(\ln(10e))^2 - (\ln(2e))^2}{4} = \boxed{2.01}$

By QUOTIENT RULE

d) $y = \frac{1}{2x} \ln 2x = \frac{\ln 2x}{2x}$

$u = \ln 2x \longrightarrow u' = \frac{1}{x}$

$v = 2x \longrightarrow v' = 2$

$\frac{dy}{dx} = \frac{\frac{2x}{x} - 2\ln 2x}{(2x)^2} \quad \left(= \frac{vu' - uv'}{v^2} \right)$

$= \frac{2 - \ln(4x^2)}{4x^2} = \frac{2 - \ln(e^4)}{e^4}$

at $x = \frac{e^2}{2}$

$= \frac{-2}{e^4}$ //

at $x = \frac{e^2}{2}$, $y = \frac{1}{e^2} \ln e^2 = \frac{2}{e^2}$ //

$\therefore y - \frac{2}{e^2} = \frac{-2}{e^4}\left(x - \frac{e^2}{2}\right)$

$\Rightarrow y = -\frac{2x}{e^4} + \frac{1}{e^2} + \frac{2}{e^2}$

$\Rightarrow \boxed{y = \frac{-2x}{e^4} + \frac{3}{e^2}}$

Q14a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \boxed{4\pi r^2}$$

b) $$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2} \quad \text{and} \quad \frac{dV}{dt} = \frac{9000\pi}{(t+81)^{5/4}}$$

$$\therefore \quad \frac{dr}{dt} = \frac{9000\cancel{\pi}}{(t+81)^{5/4} \times 4\cancel{\pi} r^2} = \boxed{\frac{2250}{r^2(t+81)^{5/4}}}$$

c) $$\frac{dr}{dt} = \frac{2250}{r^2(t+81)^{5/4}}$$

$$r^2 \frac{dr}{dt} = \frac{2250}{(t+81)^{5/4}}$$

$$\int (r^2)\,dr = 2250 \int (t+81)^{-5/4}\,dt$$

$$\Rightarrow \frac{r^3}{3} = 2250\left[\frac{(t+81)^{-\frac{1}{4}}}{-\frac{1}{4}}\right] + c$$

$$\Rightarrow \frac{r^3}{3} = \frac{-9000}{(t+81)^{1/4}} + c \quad //$$

$\underline{t=0,\ r=3}$: $9 = \dfrac{-9000}{81^{1/4}} + c$

$c = 3009$ //

$\therefore \dfrac{r^3}{3} = \dfrac{-9000}{(t+81)^{1/4}} + 3009$.

$r^3 = \dfrac{-27000}{\sqrt[4]{t+81}} + 9027$

$$\boxed{r = \left[9027 - \dfrac{27000}{\sqrt[4]{t+81}}\right]^{\frac{1}{3}}}$$

d) $\underline{t=175}$: $r = \left[9027 - \dfrac{27000}{\sqrt[4]{256}}\right]^{\frac{1}{3}} = \boxed{13.2}$

e) $\dfrac{dr}{dt} = \dfrac{2250}{r^2(t+81)^{5/4}}$ from (c).

at $t=175$, $\dfrac{dr}{dt} = \dfrac{2250}{(13.2)^2(175+81)^{5/4}} = \boxed{0.0126}$ cm/s.
$(r=13.2)$