

1. A curve  $C$  has the equation

$$x^3 + 2xy - x - y^3 - 20 = 0$$

(a) Find  $\frac{dy}{dx}$  in terms of  $x$  and  $y$ .

(5)

(b) Find an equation of the tangent to  $C$  at the point  $(3, -2)$ , giving your answer in the form  $ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers.

(2)

$$\frac{d}{dx}(x^3 + 2xy - x - y^3 - 20) = 0$$

$$3x^2 + 2x \frac{dy}{dx} + 2y - 1 - 3y^2 \frac{dy}{dx} = 0$$

$$3x^2 + 2y - 1 = (3y^2 - 2x) \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 + 2y - 1}{3y^2 - 2x}$$

$$(3, -2) \quad M_t = \frac{22}{6} = \frac{11}{3}$$

$$y + 2 = \frac{11}{3}(x - 3) \quad \Rightarrow \quad 3y + 6 = 11x - 33$$

$$11x - 3y - 39 = 0$$

2. Given that the binomial expansion of  $(1 + kx)^{-4}$ ,  $|kx| < 1$ , is

$$1 - 6x + Ax^2 + \dots$$

(a) find the value of the constant  $k$ ,

(2)

(b) find the value of the constant  $A$ , giving your answer in its simplest form.

(3)

$$(1 + kx)^{-4} = 1 + (-4)(ux) + \frac{(-4)(-5)}{2}(ux)^2$$

$$= 1 - 4kx + 10k^2x^2$$

$$\therefore \frac{k}{2} = 1.5 \quad A = \frac{10k^2}{2} = \frac{22.5}{2}$$

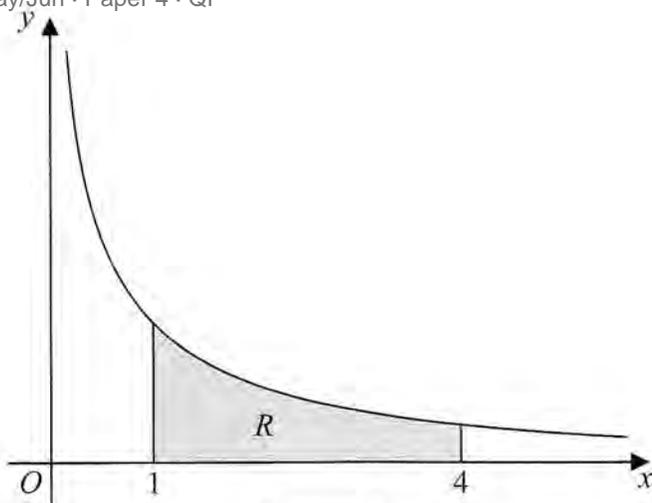


Figure 1

Figure 1 shows a sketch of part of the curve with equation  $y = \frac{10}{2x + 5\sqrt{x}}$ ,  $x > 0$

The finite region  $R$ , shown shaded in Figure 1, is bounded by the curve, the  $x$ -axis, and the lines with equations  $x = 1$  and  $x = 4$

The table below shows corresponding values of  $x$  and  $y$  for  $y = \frac{10}{2x + 5\sqrt{x}}$

$x$	1	2	3	4
$y$	1.42857	0.90326	0.68212	0.55556

- (a) Complete the table above by giving the missing value of  $y$  to 5 decimal places. (1)
- (b) Use the trapezium rule, with all the values of  $y$  in the completed table, to find an estimate for the area of  $R$ , giving your answer to 4 decimal places. (3)
- (c) By reference to the curve in Figure 1, state, giving a reason, whether your estimate in part (b) is an overestimate or an underestimate for the area of  $R$ . (1)
- (d) Use the substitution  $u = \sqrt{x}$ , or otherwise, to find the exact value of

$$\int_1^4 \frac{10}{2x + 5\sqrt{x}} dx \quad (6)$$

b)  $\frac{1}{2}(1) [1.42857 + 2(0.90326 + 0.68212) + 0.55556]$

$= \frac{2.5774}{2}$

b) over estimate, curve will lie under trapezia.

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$$u = \sqrt{x} \quad u = x^{\frac{1}{2}}$$

$$x=1 \quad u=1$$

$$x=4 \quad u=2$$

$$u^2 = x$$

$$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$2x^{\frac{1}{2}} du = dx$$

$$2u du = dx$$

$$\int_1^2 \frac{10}{2u^2 + 5u} \times 2u du$$

$$= \int_1^2 \frac{20}{2u+5} du = 10 \int_1^2 \frac{2}{2u+5} du$$

$$= 10 [\ln |2u+5|]_1^2$$

$$= 10 [\ln 9 - \ln 7] = 10 \ln \frac{9}{7}$$

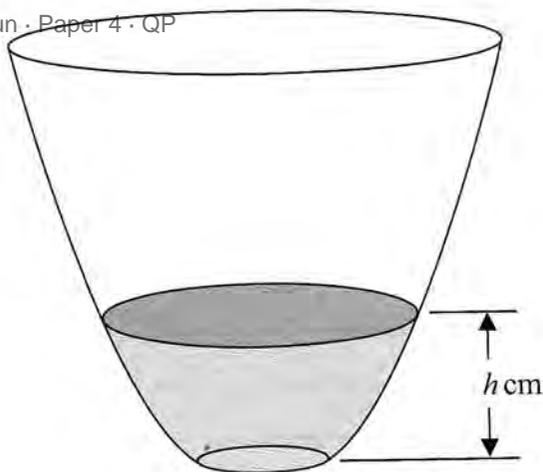


Figure 2

A vase with a circular cross-section is shown in Figure 2. Water is flowing into the vase.

When the depth of the water is  $h$  cm, the volume of water  $V$  cm<sup>3</sup> is given by

$$V = 4\pi h(h + 4), \quad 0 \leq h \leq 25$$

Water flows into the vase at a constant rate of  $80\pi$  cm<sup>3</sup>s<sup>-1</sup>

Find the rate of change of the depth of the water, in cms<sup>-1</sup>, when  $h = 6$

(5)

$$V = 4\pi h^2 + 16\pi h \quad \frac{dV}{dt} = 80\pi$$

$$\frac{dV}{dh} = 8\pi h + 16\pi$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{8\pi h + 16\pi} \times 80\pi$$

$$\frac{dh}{dt} = \frac{80\pi}{8\pi(h+2)} = \frac{10\pi}{\pi(h+2)}$$

$$h = 6 \quad \frac{dh}{dt} = \frac{10}{8}$$

$$= 1.25$$

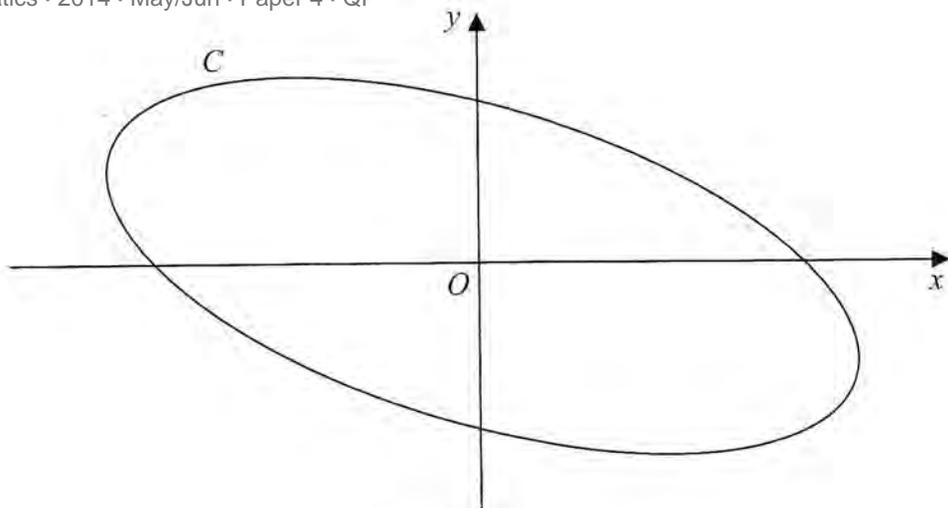


Figure 3

Figure 3 shows a sketch of the curve  $C$  with parametric equations

$$x = 4 \cos\left(t + \frac{\pi}{6}\right), \quad y = 2 \sin t, \quad 0 \leq t < 2\pi$$

(a) Show that

$$x + y = 2\sqrt{3} \cos t \quad (3)$$

(b) Show that a cartesian equation of  $C$  is

$$(x + y)^2 + ay^2 = b$$

where  $a$  and  $b$  are integers to be determined.

(2)

$$\begin{aligned} \text{a) } x &= 4 \cos t \cos \frac{\pi}{6} - 4 \sin t \sin \frac{\pi}{6} \\ x &= 2\sqrt{3} \cos t - 2 \sin t \end{aligned}$$

$$\therefore x + y = 2\sqrt{3} \cos t - 2 \sin t + 2 \sin t \quad \therefore x + y = 2\sqrt{3} \cos t$$

→

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$$(x+y)^2 = (2\sqrt{3} \cos t)^2 = 12 \cos^2 t$$

$$y^2 = 4 \sin^2 t \quad \Rightarrow \quad 4 - y^2 = 4 - 4 \sin^2 t$$

$$\Rightarrow \quad 4 - y^2 = 4 \cos^2 t$$

$$\rightarrow \quad 12 - 3y^2 = 12 \cos^2 t$$

$$\therefore (x+y)^2 = 12 - 3y^2 \quad \therefore (x+y)^2 + 3y^2 = 12$$

↷

$$\int x e^{4x} dx \quad (3)$$

(ii) Find

$$\int \frac{8}{(2x-1)^3} dx, \quad x > \frac{1}{2} \quad (2)$$

(iii) Given that  $y = \frac{\pi}{6}$  at  $x = 0$ , solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y \quad (7)$$

$$\begin{aligned} \text{i) } u &= x & v &= \frac{1}{4} e^{4x} &= \frac{1}{4} x e^{4x} - \frac{1}{4} \int e^{4x} dx \\ u' &= 1 & v' &= e^{4x} &= \frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + C \end{aligned}$$

$$\begin{aligned} \text{ii) } \int 8(2x-1)^3 dx &= -2(2x-1)^2 + C \\ u &= (2x-1)^2 & x-2 & \\ \frac{du}{dx} &= -2(2x-1)^3 \times 2 & \uparrow & \\ &= -4(2x-1)^3 & x-2 & \end{aligned}$$

$$\text{iii) } \frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y$$

$$\int \frac{1}{\operatorname{cosec} 2y \operatorname{cosec} y} dy = \int e^x dx$$

$$\int \sin 2y \sin y dy = e^x + C$$

$$u = (\sin y)^3 \times \frac{2}{3}$$

$$\int 2 \sin^2 y \cos y dy = e^x + C$$

$$\frac{du}{dx} = 3 \sin^2 y \cos y \times \frac{2}{3}$$

$$\frac{2}{3} \sin^3 y = e^x + C \quad x=0, y=\frac{\pi}{6} \quad \frac{2}{3} \left(\frac{1}{2}\right)^3 = 1 + C \quad \therefore C = -\frac{11}{12}$$

$$\frac{2}{3} \sin^3 y = e^x - \frac{11}{12}$$

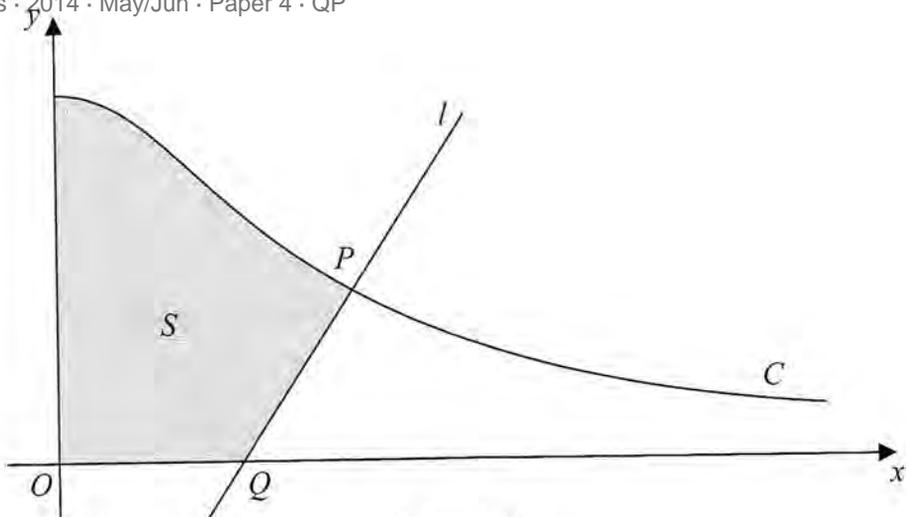


Figure 4

Figure 4 shows a sketch of part of the curve  $C$  with parametric equations

$$x = 3 \tan \theta, \quad y = 4 \cos^2 \theta, \quad 0 \leq \theta < \frac{\pi}{2}$$

The point  $P$  lies on  $C$  and has coordinates  $(3, 2)$ .

The line  $l$  is the normal to  $C$  at  $P$ . The normal cuts the  $x$ -axis at the point  $Q$ .

(a) Find the  $x$  coordinate of the point  $Q$ .

(6)

The finite region  $S$ , shown shaded in Figure 4, is bounded by the curve  $C$ , the  $x$ -axis, the  $y$ -axis and the line  $l$ . This shaded region is rotated  $2\pi$  radians about the  $x$ -axis to form a solid of revolution.

(b) Find the exact value of the volume of the solid of revolution, giving your answer in the form  $p\pi + q\pi^2$ , where  $p$  and  $q$  are rational numbers to be determined.

[You may use the formula  $V = \frac{1}{3}\pi r^2 h$  for the volume of a cone.]

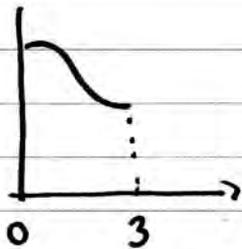
(9)

$$\begin{aligned}
 y &= 4 \cos^2 \theta & x &= 3 \tan \theta & \frac{dy}{dx} &= \frac{8 \sin \theta \cos \theta}{3 \sec^2 \theta} \\
 \frac{dy}{d\theta} &= -8 \sin \theta \cos \theta & \frac{dx}{d\theta} &= 3 \sec^2 \theta & \frac{dy}{dx} &= \frac{-8 \sin \theta \cos^3 \theta}{3} \\
 \text{at } P \quad x &= 3 & 3 &= 3 \tan \theta & & \\
 \tan \theta &= 1 & \theta &= \frac{\pi}{4} & \therefore M_v &= \frac{8 \left(\frac{\sqrt{2}}{2}\right) \left(\frac{\sqrt{2}}{2}\right)^3}{3} = \frac{8 \times 4}{3 \times 16} = \frac{-2}{3}
 \end{aligned}$$

$$M_n = +\frac{3}{2} \quad y = 2 = \frac{3}{2}(x-3)$$

$$y=0 \Rightarrow x-3 = -\frac{4}{3} \therefore x = \frac{5}{3}$$

b)



$$V = \pi \int_{0=x}^{3=x} y^2 \frac{dx}{d\theta} d\theta$$

$$x=3 \quad \theta = \frac{\pi}{4}$$

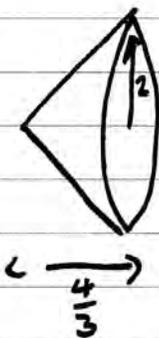
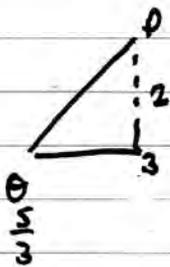
$$x=0 \quad \theta = 0$$

$$V = \pi \int_{0=x}^{3=x} 16 \cos^4 \theta \times 3 \sec^2 \theta d\theta$$

$$V = \pi 48 \int_{0=x}^{3=x} \cos^2 \theta d\theta = 48\pi \int_0^{\frac{\pi}{4}} \left( \frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$V = 24\pi \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta) d\theta$$

$$V = 24\pi \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} = 24\pi \left[ \left( \frac{\pi}{4} + \frac{1}{2} \right) - (0) \right] = 6\pi^2 + 12\pi$$



$$V = \frac{1}{3} \pi (2)^2 \left( \frac{4}{3} \right) = \frac{16}{9} \pi$$

$$\therefore S = 62\pi^2 + 12\pi - \frac{16}{9}\pi$$

$$S = 62\pi^2 + \frac{92}{9}\pi$$

8. Relative to a fixed origin  $O$ , the point  $A$  has position vector  $\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$

and the point  $B$  has position vector  $\begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$

The line  $l_1$  passes through the points  $A$  and  $B$ .

(a) Find the vector  $\overrightarrow{AB}$ . (2)

(b) Hence find a vector equation for the line  $l_1$ . (1)

The point  $P$  has position vector  $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

Given that angle  $PBA$  is  $\theta$ ,

(c) show that  $\cos \theta = \frac{1}{3}$ . (3)

The line  $l_2$  passes through the point  $P$  and is parallel to the line  $l_1$ .

(d) Find a vector equation for the line  $l_2$ . (2)

The points  $C$  and  $D$  both lie on the line  $l_2$ .

Given that  $AB = PC = DP$  and the  $x$  coordinate of  $C$  is positive,

(e) find the coordinates of  $C$  and the coordinates of  $D$ . (3)

(f) find the exact area of the trapezium  $ABCD$ , giving your answer as a simplified surd. (4)

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$$a) \vec{a} = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} \quad \vec{AB} = \vec{b} - \vec{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$b) l_1 = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$c) \vec{p} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \quad \vec{BP} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} \quad |\vec{BP}| = \sqrt{1^2 + 1^2 + 5^2} = \sqrt{27} = 3\sqrt{3}$$

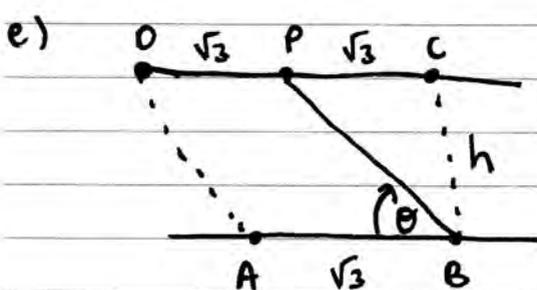
$$\vec{BA} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$|\vec{BA}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\vec{BP} \cdot \vec{BA} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -1 - 1 + 5 = 3$$

$$\therefore \cos \theta = \frac{(\vec{BP} \cdot \vec{BA})}{(|\vec{BP}| |\vec{BA}|)} = \frac{3}{(3\sqrt{3})(\sqrt{3})} = \frac{3}{9} = \frac{1}{3} \neq$$

$$d) \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

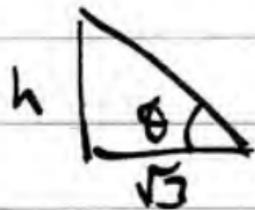


$$\therefore \vec{C} = \vec{p} + \vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

$$\vec{D} = \vec{p} - \vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

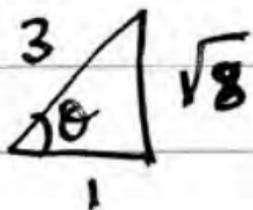
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$$A) \text{ Area} = (\sqrt{3} + 2\sqrt{3}) \times \frac{1}{2} h$$



$$h = \sqrt{3} \tan \theta$$

$$\cos \theta = \frac{1}{3}$$



$$\therefore \tan \theta = \sqrt{8}$$

$$\therefore h = \sqrt{8} \sqrt{3}$$

$$\therefore \text{Area} = 3\sqrt{3} \times \frac{\sqrt{8} \sqrt{3}}{2}$$

$$= \frac{9}{2} \times 2\sqrt{2} = 9\sqrt{2}$$