

Mathematics · 2016 · Jan · Paper 4 · QP

$$f(x) = (3 - 2x)^{-4}, \quad |x| < \frac{3}{2}$$

1.

Find the binomial expansion of  $f(x)$ , in ascending powers of  $x$ , up to and including the term in  $x^2$ , giving each coefficient as a simplified fraction.

(4)

$$f(x) = \left[ 3 \left( 1 - \frac{2x}{3} \right) \right]^{-4} = \frac{1}{3^4} \left( 1 - \frac{2x}{3} \right)^{-4}$$

$$\approx \frac{1}{81} \left[ 1 + (-4) \left( -\frac{2x}{3} \right) + (-4)(-5) \left( -\frac{2x}{3} \right)^2 \cdot \frac{1}{2} \right]$$

$$= \frac{1}{81} \left[ 1 + \frac{8x}{3} + \frac{40x^2}{9} \right]$$

$$= \frac{1}{81} + \frac{8x}{243} + \frac{40x^2}{729}$$

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2. (a) Show that

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

may be expressed in the form  $\operatorname{cosec}^2 x - \operatorname{cosec} x + k = 0$ , where  $k$  is a constant. (1)

(b) Hence solve for  $0 \leq x < 360^\circ$

$$\cot^2 x - \operatorname{cosec} x - 11 = 0$$

Give each solution in degrees to one decimal place.

(Solutions based entirely on graphical or numerical methods are not acceptable.) (5)

$$a) \cot^2 x \equiv \operatorname{cosec}^2 x - 1$$

$$\Rightarrow \operatorname{cosec}^2 x - 1 - \operatorname{cosec} x - 11 = 0$$

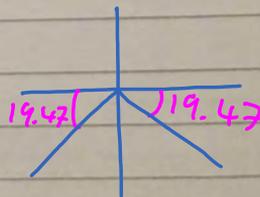
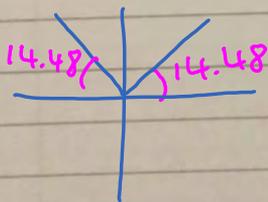
$$\operatorname{cosec}^2 x - \operatorname{cosec} x - 12 = 0$$

$$b) (\operatorname{cosec} x - 4)(\operatorname{cosec} x + 3) = 0$$

$$\operatorname{cosec} x = 4 \quad \text{or} \quad \operatorname{cosec} x = -3$$

$$\sin x = \frac{1}{4}$$

$$\sin x = -\frac{1}{3}$$



$$\theta = 14.5, 165.5, 199.5, 340.5$$

3. A curve  $C$  has equation

$$3^x + 6y = \frac{3}{2}xy^2$$

Find the exact value of  $\frac{dy}{dx}$  at the point on  $C$  with coordinates  $(2, 3)$ . Give your answer in the form  $\frac{a + \ln b}{8}$ , where  $a$  and  $b$  are integers. (7)

$$\text{Let } z = 3^x$$

$$\ln z = x \ln 3$$

$$\frac{1}{z} \frac{dz}{dx} = \ln 3$$

$$\frac{dz}{dx} = 3^x \ln 3$$

$$\Rightarrow 3^x \ln 3 + 6 \frac{dy}{dx} = \frac{3}{2}x - 2y \frac{dy}{dx} + \frac{3}{2}y^2$$

$$\frac{dy}{dx} = \frac{3^x \ln 3 - \frac{3}{2}y^2}{3xy - 6}$$

$$\text{At } (2, 3),$$

$$\frac{dy}{dx} = \frac{3^2 \ln 3 - \frac{3}{2}3^2}{3(2)(3) - 6} = \frac{\cancel{3}9 \ln 3 - \cancel{2}7 \cdot \frac{3}{2}}{\cancel{12}4}$$

$$= \frac{-9 + 6 \ln 3}{8}$$

$$= \frac{-9 + \ln 729}{8}$$

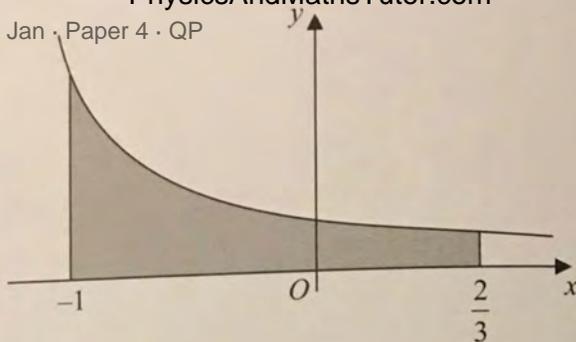


Figure 1

The curve  $C$  with equation  $y = \frac{2}{(4+3x)}$ ,  $x > -\frac{4}{3}$  is shown in Figure 1

The region bounded by the curve, the  $x$ -axis and the lines  $x = -1$  and  $x = \frac{2}{3}$ , is shown shaded in Figure 1

This region is rotated through 360 degrees about the  $x$ -axis.

(a) Use calculus to find the exact value of the volume of the solid generated.

(5)

$$V = \pi \int_{-1}^{2/3} y^2 dx$$

$$= 4\pi \int (4+3x)^{-2} dx$$

$$= \frac{4\pi}{(-1)(3)} \left[ \frac{1}{4+3x} \right]_{-1}^{2/3}$$

$$= -\frac{4\pi}{3} \left[ \left(\frac{1}{6}\right) - (1) \right] = \frac{10\pi}{9}$$

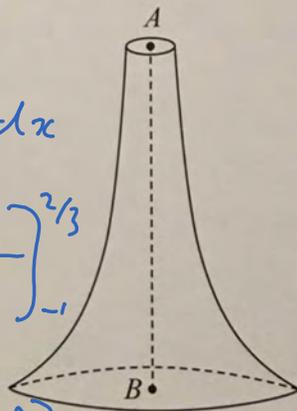


Figure 2

Figure 2 shows a candle with axis of symmetry  $AB$  where  $AB = 15$  cm.  $A$  is a point at the centre of the top surface of the candle and  $B$  is a point at the centre of the base of the candle. The candle is geometrically similar to the solid generated in part (a).

(b) Find the volume of this candle.

(2)

$$\frac{V}{\frac{10\pi}{9}} = \left(\frac{15}{5/3}\right)^3$$

$$V = 810\pi$$

$$f(x) = -x^3 + 4x^2 - 6$$

(a) Show that the equation  $f(x) = 0$  has a root between  $x = 1$  and  $x = 2$

(2)

(b) Show that the equation  $f(x) = 0$  can be rewritten as

$$x = \sqrt{\left(\frac{6}{4-x}\right)}$$

(2)

(c) Starting with  $x_1 = 1.5$  use the iteration  $x_{n+1} = \sqrt{\left(\frac{6}{4-x_n}\right)}$  to calculate the values of  $x_2$ ,  $x_3$  and  $x_4$  giving all your answers to 4 decimal places.

(3)

(d) Using a suitable interval, show that 1.572 is a root of  $f(x) = 0$  correct to 3 decimal places.

(2)

$$a) f(1) = -(1)^3 + 4(1)^2 - 6 = -3 < 0$$

$$f(2) = -(2)^3 + 4(2)^2 - 6 = 2 > 0$$

There's a change of sign ...

$$b) 0 = -x^3 + 4x^2 - 6$$

$$6 = x^2(4-x)$$

$$x = \sqrt{\frac{6}{4-x}}$$

$$c) x_1 = 1.5$$

$$x_2 = 1.5492$$

$$x_3 = 1.5647$$

$$x_4 = 1.5696$$

$$d) f(1.5715) = -2.5 \times 10^{-3} < 0$$

$$f(1.5725) = 2.6 \times 10^{-3} > 0$$

There's a sign change ...

6. A hot piece of metal is dropped into a cool liquid. As the metal cools, its temperature  $T$  degrees Celsius,  $t$  minutes after it enters the liquid, is modelled by

$$T = 300e^{-0.04t} + 20, \quad t \geq 0$$

- (a) Find the temperature of the piece of metal as it enters the liquid. (1)
- (b) Find the value of  $t$  for which  $T = 180$ , giving your answer to 3 significant figures. (4)
- (Solutions based entirely on graphical or numerical methods are not acceptable.)
- (c) Show, by differentiation, that the rate, in degrees Celsius per minute, at which the temperature of the metal is changing, is given by the expression

$$\frac{20 - T}{25}$$

(3)

a) At  $t = 0$ ,  $T = 300 + 20 = 320^\circ\text{C}$

b)  $180 = 300e^{-0.04t} + 20$

$$8/15 = e^{-0.04t}$$

$$t = \frac{-\ln(8/15)}{0.04} = 15.7 \text{ min. (3sf)}$$

c)  $\frac{dT}{dt} = -0.04(300)e^{-0.04t}$

$$= -\frac{1}{25}(T - 20)$$

$$= \frac{20 - T}{25}$$

7.

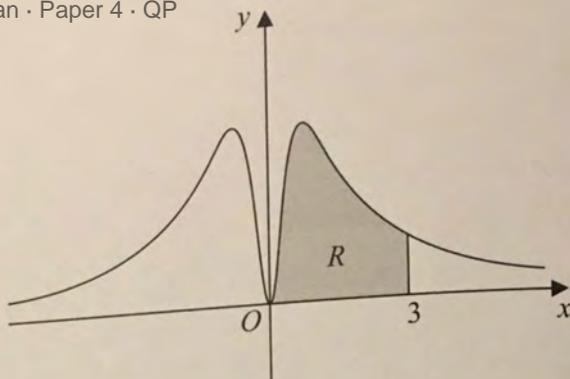


Figure 3

Figure 3 shows part of the curve  $C$  with equation

$$y = \frac{3 \ln(x^2 + 1)}{(x^2 + 1)}, \quad x \in \mathbb{R}$$

- (a) Find  $\frac{dy}{dx}$  (2)
- (b) Using your answer to (a), find the exact coordinates of the stationary point on the curve  $C$  for which  $x > 0$ . Write each coordinate in its simplest form. (5)

The finite region  $R$ , shown shaded in Figure 3, is bounded by the curve  $C$ , the  $x$ -axis and the line  $x = 3$

- (c) Complete the table below with the value of  $y$  corresponding to  $x = 1$

$x$	0	1	2	3
$y$	0	$\frac{3}{2} \ln 2$	$\frac{3}{5} \ln 5$	$\frac{3}{10} \ln 10$

(1)

- (d) Use the trapezium rule with all the  $y$  values in the completed table to find an approximate value for the area of  $R$ , giving your answer to 4 significant figures. (3)

$$\begin{aligned}
 a) \frac{dy}{dx} &= \frac{3(x^2+1) \cdot 2x(x^2+1)^{-1} - 3 \ln(x^2+1) \cdot 2x}{(x^2+1)^2} \\
 &= \frac{6x}{(x^2+1)^2} [1 - \ln(x^2+1)]
 \end{aligned}$$

$$b) 0 = \frac{6x}{(x^2+1)^2} [1 - \ln(x^2+1)]$$

$$x=0 \quad \text{or} \quad 1 - \ln(x^2+1) = 0$$

$$\ln(x^2+1) = 1$$

$$x^2+1 = e$$

$$x = \pm\sqrt{e-1}$$

$$\text{When } x = \sqrt{e-1}, y = \frac{3(1)}{e}$$

$$\therefore \text{Stationary point is } (\sqrt{e-1}, 3e^{-1})$$

$$d) R = \frac{1}{2} (1) \left[ 0 + \frac{3}{10} \ln 10 + 2 \left( \frac{3}{2} \ln 2 + \frac{3}{5} \ln 5 \right) \right]$$

$$= 2.351 \quad (4 \text{ sf})$$

8. Mathematics · 2016 · Jan · Paper 4 · QP  $f(\theta) = 9\cos^2\theta + \sin^2\theta$

(a) Show that  $f(\theta) = a + b\cos 2\theta$ , where  $a$  and  $b$  are integers which should be found. (3)

(b) Using your answer to part (a) and integration by parts, find the exact value of

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta) d\theta \quad (6)$$

$$a) \cos 2\theta \equiv 2\cos^2\theta - 1 \equiv 1 - 2\sin^2\theta$$

$$f(\theta) = 9\left(\frac{\cos 2\theta + 1}{2}\right) + \frac{1 - \cos 2\theta}{2}$$

$$= 5 + 4\cos 2\theta$$

$$b) \begin{array}{l|l} u = \theta^2 & v' = f(\theta) \\ u' = 2\theta & v = 5\theta + 2\sin 2\theta \end{array} \quad \left| \quad \begin{array}{l} u = \theta & v' = \sin 2\theta \\ u' = 1 & v = -\frac{1}{2}\cos 2\theta \end{array} \right.$$

$$\int_0^{\frac{\pi}{2}} \theta^2 f(\theta) d\theta = \left[ 5\theta^3 + 2\theta \sin 2\theta \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (10\theta^2 + 4\theta \sin 2\theta) d\theta$$

$$= \frac{5\pi^3}{8} - 10\left[\frac{\theta^3}{3}\right]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} \theta \sin 2\theta d\theta$$

$$= \frac{5\pi^3}{24} + 2\left[\theta \cos 2\theta\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\cos 2\theta d\theta$$

$$= \frac{5\pi^3}{24} - \pi - \left[\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{5\pi^3}{24} - \pi$$

9. (a) Express  $\frac{3x^2 - 4}{x^2(3x - 2)}$  in partial fractions.

(4)

(b) Given that  $x > \frac{2}{3}$ , find the general solution of the differential equation

$$x^2(3x - 2) \frac{dy}{dx} = y(3x^2 - 4)$$

Give your answer in the form  $y = f(x)$ .

(6)

$$a) \frac{3x^2 - 4}{x^2(3x - 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{3x - 2}$$

$$3x^2 - 4 = Ax(3x - 2) + B(3x - 2) + Cx^2$$

$$x = 0: -4 = -2B$$

$$B = 2$$

$$\text{Coef. } x: 0 = -2A + 3B$$

$$A = 3$$

$$\text{Coef. } x^2: 3 = 3A + C$$

$$C = -6$$

$$\Rightarrow \frac{3x^2 - 4}{x^2(3x - 2)} = \frac{2}{x^2} + \frac{3}{x} - \frac{6}{3x - 2}$$

$$b) \int \frac{1}{y} dy = \int \frac{3x^2 - 4}{x^2(3x - 2)} dx$$

$$\ln y + C = -2x^{-1} + 3 \ln|x| - 2 \ln|3x - 2|$$

$$\ln y = \ln\left(\frac{Ax^3}{3x - 2}\right) - \frac{2}{x}$$

$$y = \frac{Ax^3}{3x - 2} \cdot e^{-2/x}$$

10. (a) Express  $3 \sin 2x + 5 \cos 2x$  in the form  $R \sin(2x + \alpha)$ , where  $R > 0$  and  $0 < \alpha < \frac{\pi}{2}$

Give the exact value of  $R$  and give the value of  $\alpha$  to 3 significant figures.

(3)

(b) Solve, for  $0 < x < \pi$ ,

$$3 \sin 2x + 5 \cos 2x = 4$$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

$$g(x) = 4(3 \sin 2x + 5 \cos 2x)^2 + 3$$

(c) Using your answer to part (a) and showing your working,

(i) find the greatest value of  $g(x)$ ,

(ii) find the least value of  $g(x)$ .

(4)

$$\begin{aligned} a) R \sin(2x + \alpha) &= R \sin 2x \cos \alpha + R \cos 2x \sin \alpha \\ &= 3 \sin 2x + 5 \cos 2x \end{aligned}$$

$$\begin{aligned} 5 &= R \sin \alpha \\ 3 &= R \cos \alpha \end{aligned}$$

$$\tan \alpha = \frac{5}{3}$$

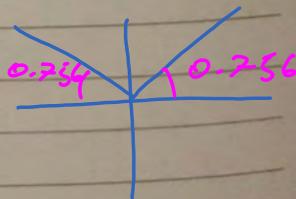
$$\alpha = 1.03$$

$$\begin{aligned} R &= \sqrt{5^2 + 3^2} \\ &= \sqrt{34} \end{aligned}$$

$$b) \sqrt{34} \sin(2x + 1.03) = 4$$

$$2x + 1.03 = 2.39, 7.04$$

$$x = 0.68, 3.01$$



$$c) g(x) = 4 \left[ \underbrace{\sqrt{34} \sin(2x + 1.03)}_{\text{max} = 34, \text{min} = 0} \right]^2 + 3$$

$$i) 4(34) + 3 = 139$$

$$ii) 4(0) + 3 = 3$$

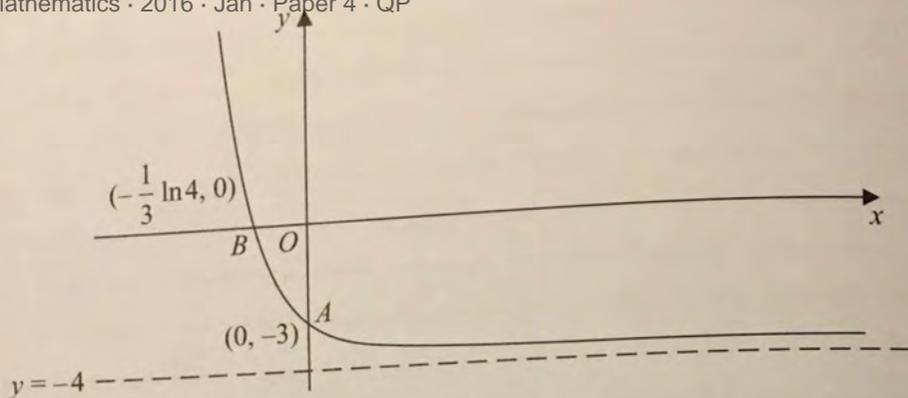


Figure 4

Figure 4 shows a sketch of part of the curve with equation  $y = f(x)$ ,  $x \in \mathbb{R}$

The curve meets the coordinate axes at the points  $A(0, -3)$  and  $B(-\frac{1}{3} \ln 4, 0)$  and the curve has an asymptote with equation  $y = -4$

In separate diagrams, sketch the graph with equation

(a)  $y = |f(x)|$  (4)

(b)  $y = 2f(x) + 6$  (3)

On each sketch, give the exact coordinates of the points where the curve crosses or meets the coordinate axes and the equation of any asymptote.

Given that

$$f(x) = e^{-3x} - 4, \quad x \in \mathbb{R}$$

$$g(x) = \ln\left(\frac{1}{x+2}\right), \quad x > -2$$

(c) state the range of  $f$ ,

$$f(x) > -4$$

$$y + 4 = e^{-3x}$$

$$x = -\frac{\ln(y+4)}{3}$$

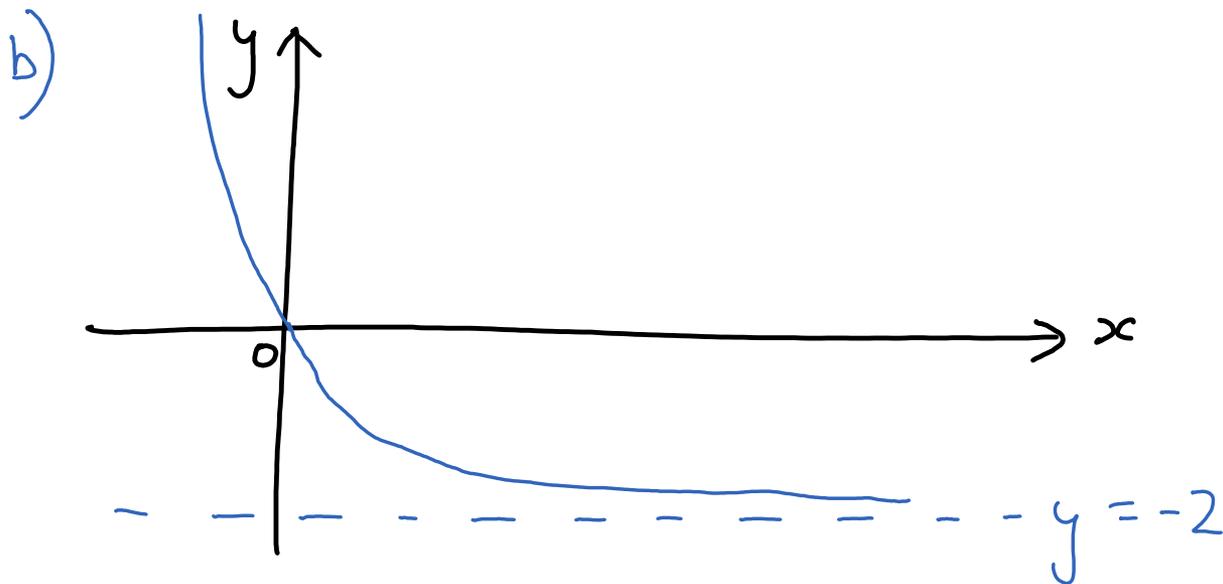
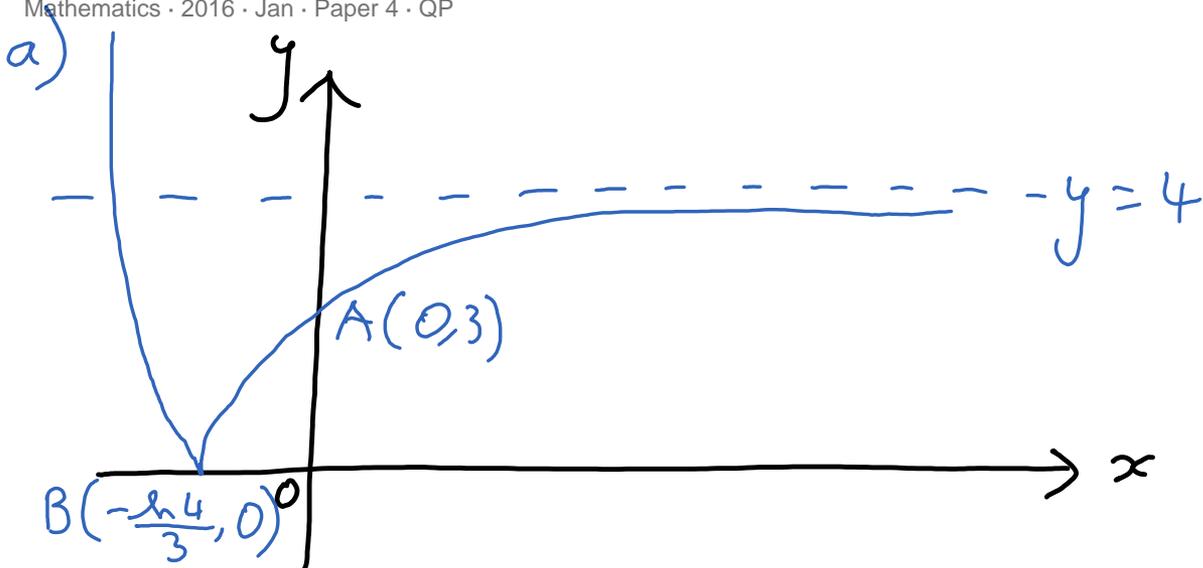
(d) find  $f^{-1}(x)$ ,

$$f^{-1}(x) = -\frac{\ln(x+4)}{3}$$

(e) express  $fg(x)$  as a polynomial in  $x$ .

$$\begin{aligned} fg(x) &= e^{-3 \ln\left(\frac{1}{x+2}\right)} - 4 \\ &= (x+2)^3 - 4 \end{aligned}$$

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12. With respect to a fixed origin  $O$ , the lines  $l_1$  and  $l_2$  are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 12 \\ -4 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ 2 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

where  $\lambda$  and  $\mu$  are scalar parameters.

(a) Show that  $l_1$  and  $l_2$  meet, and find the position vector of their point of intersection  $A$ . (6)

(b) Find, to the nearest  $0.1^\circ$ , the acute angle between  $l_1$  and  $l_2$ . (3)

The point  $B$  has position vector  $\begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$ .

(c) Show that  $B$  lies on  $l_1$ . (1)

(d) Find the shortest distance from  $B$  to the line  $l_2$ , giving your answer to 3 significant figures. (4)

$$\begin{aligned} \text{a) } 12 + 5\lambda &= 2 & (1) \\ -4 - 4\lambda &= 2 + 6\mu & (2) \\ 5 + 2\lambda &= 3\mu & (3) \end{aligned}$$

$$(1): \lambda = -2$$

$$(3): 5 + 2(-2) = 3\left(\frac{1}{3}\right)$$

$$\begin{aligned} (2): \mu &= \frac{-6 - 4(-2)}{6} \\ &= \frac{1}{3} \end{aligned}$$

$$5 - 4 = 1$$

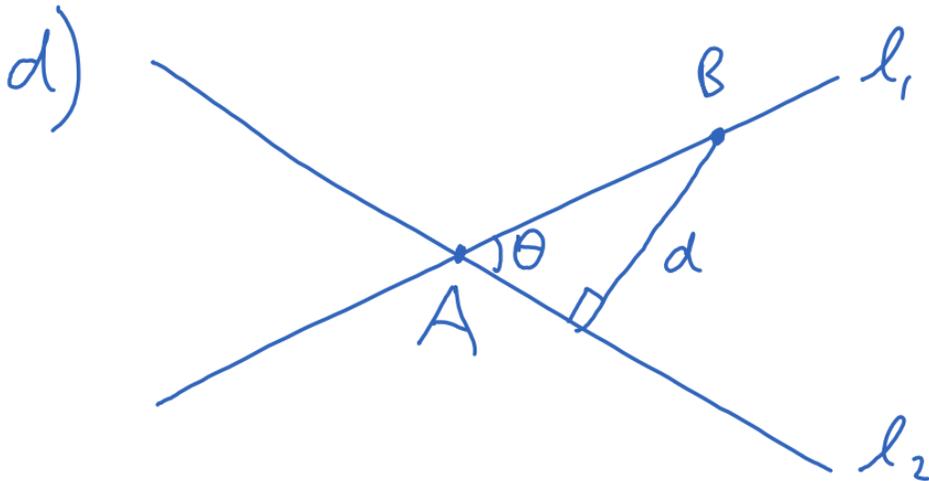
$$\text{When } \mu = \frac{1}{3}, \mathbf{r}_A = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

$$\text{b) } \cos \theta = \frac{|5(0) - 4(6) + 2(3)|}{\sqrt{5^2 + 4^2 + 2^2} \sqrt{6^2 + 3^2}}$$

$$\theta = 66.4^\circ$$

c) When  $\lambda = -1$ ,

$$\mathbf{r} = \begin{pmatrix} 12 - 5 \\ -4 + 4 \\ 5 - 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}$$



$$d = AB \sin \theta$$

$$= \sqrt{(7-2)^2 + (4-0)^2 + (3-1)^2} \sin 66.4$$

$$= 6.15 \text{ (3sf)}$$

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 13. A curve  $C$  has parametric equations

$$x = 6 \cos 2t, \quad y = 2 \sin t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

(a) Show that  $\frac{dy}{dx} = \lambda \operatorname{cosec} t$ , giving the exact value of the constant  $\lambda$ . (4)

(b) Find an equation of the normal to  $C$  at the point where  $t = \frac{\pi}{3}$

Give your answer in the form  $y = mx + c$ , where  $m$  and  $c$  are simplified surds. (6)

The cartesian equation for the curve  $C$  can be written in the form

$$x = f(y), \quad -k < y < k$$

where  $f(y)$  is a polynomial in  $y$  and  $k$  is a constant.

(c) Find  $f(y)$ . (3)

(d) State the value of  $k$ . (1)

$$a) \frac{dx}{dt} = -12 \sin 2t \quad \frac{dy}{dt} = 2 \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2 \cos t}{-12 \sin 2t} = \frac{2 \cos t}{-24 \sin t \cos t}$$

$$= -\frac{1}{12} \operatorname{cosec} t$$

$$b) \text{ At } t = \frac{\pi}{3}, \quad x = 6 \cos\left(2 \cdot \frac{\pi}{3}\right) = -3$$

$$y = 2 \sin\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\frac{dy}{dx} = -\frac{1}{12} \cdot \frac{1}{\sin \frac{\pi}{3}} = -\frac{1}{6\sqrt{3}}$$

$$\Rightarrow m_n = 6\sqrt{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{3} = 6\sqrt{3}(x + 3)$$

$$y = 6\sqrt{3}x + 19\sqrt{3}$$

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$$c) x = 6 \cos^2 t = 6(1 - 2 \sin^2 t)$$

$$= 6 - 12 \sin^2 t$$

$$= 6 - 3y^2$$

$$d) k = 2$$