C4 June 2017 (MA)

Q1a) $x = 3t - 4$ $\quad\quad$ $y = 5 - 6t^{-1}$

$\frac{dx}{dt} = 3$ $\quad\quad$ $\frac{dy}{dt} = 6t^{-2}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t^{-2}}{3} = \boxed{\frac{2}{t^2}}$$

b) at $t = \frac{1}{2}$: $x = \frac{3}{2} - 4 = -\frac{5}{2}$, $y = 5 - \frac{6}{\frac{1}{2}} = -7$ $\quad P\left(-\frac{5}{2}, -7\right)$

at P, $\frac{dy}{dx} = \frac{2}{(\frac{1}{2})^2} = 8 \therefore m_{tangent} = 8.$

$\Rightarrow y - (-7) = 8(x - (-\frac{5}{2}))$

$\Rightarrow y = 8x + 20 - 7$

$\Rightarrow \boxed{y = 8x + 13}$

c) $y = 5 - \frac{6}{t}$

$x = 3t - 4$

$\therefore \left(\frac{x+4}{3} = t\right) \xrightarrow{\text{sub into } y} y = 5 - \frac{6}{\frac{x+4}{3}}$

$\therefore y = 5 - \frac{18}{x+4}$

$$y = \frac{5(x+4)}{x+4} - \frac{18}{x+4} = \frac{5x+20-18}{x+4}$$

$$\boxed{y = \frac{5x+2}{x+4}} \qquad a = 5, \quad b = 2$$

Q2a) $(2+kx)^{-3} = 2^{-3}\left(1+\frac{kx}{2}\right)^{-3} = \frac{1}{8}\left(1+\frac{kx}{2}\right)^{-3}$

$\therefore \boxed{A = \frac{1}{8}}$

b) $\frac{1}{8}\left(1+\frac{kx}{2}\right)^{-3} = \frac{1}{8}\left[1 - \frac{3kx}{2} + \frac{-3(-4)}{2}\left(\frac{kx}{2}\right)^2\right]$

$\begin{bmatrix} n = -3 \\ x = \frac{kx}{2} \end{bmatrix} \quad = \frac{1}{8}\left[1 - \frac{3kx}{2} + \frac{3k^2x^2}{2}\right]$

$$\simeq \frac{1}{8} - \frac{3kx}{16} + \frac{3k^2x^2}{16} \quad //$$

$B = -\frac{3k}{16}$ and $\frac{3k^2}{16} = \frac{243}{16}$

$\Rightarrow 3k^2 = 243$

$\Rightarrow k^2 = \frac{243}{3} = 81$

$\Rightarrow k = \sqrt{81} = \boxed{9} \quad (k > 0)$

c) $\therefore B = -\frac{3}{16}(9) = \boxed{-\frac{27}{16}}$

Q3a)

x	0.2
y	1.86254

b) $h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$

$$\text{Area} \simeq \frac{1}{2} \times \frac{1}{5}\left[2 + 1.27165 + 2(1.41994 + 1.56981 + 1.7183 + 1.86254)\right]$$

$$\simeq \boxed{1.6413}$$

c) $$\text{Area} = \int_0^1 \left(\frac{6}{e^x+2}\right) dx$$

$$\left[u = e^x \quad \frac{du}{dx} = e^x \quad \therefore dx = e^{-x}\,du \right]$$

$$= \int_1^e \left[\frac{6}{u+2} \times e^{-x}\right] du$$

x	u
0	1
1	e

$$= \int_1^e \left[\frac{6}{u+2} \times \frac{1}{u}\right] du$$

$$\left(e^{-x} = \frac{1}{u}\right)$$

$$= \int_1^e \left[\frac{6}{u(u+2)}\right] du \quad // \square$$

d) $R = 6\int_1^e \left[\frac{1}{u(u+2)}\right] du$ => By partial fractions...

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$$

$$1 = A(u+2) + B(u)$$

$\underline{u=0}$: $1 = 2A \quad \therefore \quad A = \frac{1}{2}$ //

$\underline{u=1}$: $1 = 3A + B \quad \therefore \quad B = 1 - \frac{3}{2} = -\frac{1}{2}$ //

$$R = 6\int_1^e \left[\frac{\frac{1}{2}}{u} - \frac{\frac{1}{2}}{u+2}\right] du$$

$$= 6\left[\frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2|\right]_1^e$$

$$= 6\left[\frac{1}{2}\ln\left|\frac{u}{u+2}\right|\right]_1^e = 6\left[\frac{1}{2}\ln\frac{e}{e+2}\right] - 6\left[\frac{1}{2}\ln\frac{1}{3}\right]$$

$$= 3\ln\frac{e}{e+2} - 3\ln\frac{1}{3} = \boxed{3\ln\left(\frac{3e}{e+2}\right)} //$$

or equivalent.
(many forms)

Q4a) $\frac{d}{dx}\left(4x^2 - y^3 - 4xy + 2^y = 0\right)$

$\Rightarrow 8x - 3y^2\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} + (2^y \ln 2)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx}\left(2^y \ln 2 - 4x - 3y^2\right) = 4y - 8x$

$\Rightarrow \frac{dy}{dx} = \frac{4y - 8x}{2^y \ln 2 - 4x - 3y^2}$ //

at P, $\frac{dy}{dx} = \frac{4(4) - 8(-2)}{2^4 \ln 2 - 4(-2) - 3(16)} = \boxed{\frac{32}{16\ln 2 - 40}}$

b) at normal to P, $m = \frac{-(16\ln 2 - 40)}{32} = \frac{40 - 16\ln 2}{32}$

$\Rightarrow y - 4 = \frac{40 - 16\ln 2}{32}(x + 2)$

$\underline{x = 0}$: $y = 4 + \frac{2(40 - 16\ln 2)}{32}$

$y = 4 + \frac{40 - 16\ln 2}{16}$

$y = \frac{64 + 40 - 16\ln 2}{16} = \boxed{\frac{13}{2} - \ln 2}$

Q5) $V = \pi \int_0^{\ln 4} (y^2)\,dx = \pi \int_0^{\ln 4} \left[(e^x + 2e^{-x})(e^x + 2e^{-x}) \right] dx$

$= \pi \int_0^{\ln 4} \left[e^{2x} + 4 + 4e^{-2x} \right] dx$

$= \pi \left[\frac{1}{2} e^{2x} + 4x - 2e^{-2x} \right]_0^{\ln 4}$

$= \pi \left[\frac{1}{2} e^{2\ln 4} + 4\ln 4 - 2e^{-2\ln 4} \right] - \pi \left[\frac{1}{2} - 2 \right]$

$= \pi \left[\frac{1}{2} e^{\ln 16} + 4\ln 4 - 2e^{\ln \frac{1}{16}} \right] + \frac{3\pi}{2}$

$= \pi \left[8 + 4\ln 4 - \frac{2}{16} + \frac{3}{2} \right] = \pi \left[\frac{75}{8} + 4\ln 4 \right]$

$= \boxed{\pi \left[\frac{75}{8} + \ln 256 \right]}$

Q6a) $\begin{pmatrix} 4-\lambda \\ 28-5\lambda \\ 4+\lambda \end{pmatrix} = \begin{pmatrix} 5+3\mu \\ 3 \\ 1-4\mu \end{pmatrix}$ — ① ② ③

② : $28 - 5\lambda = 3$
$25 = 5\lambda \quad \therefore \lambda = 5$

$\underline{\lambda = 5}$: $X = \begin{pmatrix} 4-5 \\ 28-25 \\ 4+5 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix}}$

b) using dir. vectors : $\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = -3 - 4 = -7$

$\left|\begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}\right| = \sqrt{1^2+5^2+1^2} = 3\sqrt{3}$

$\left|\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}\right| = \sqrt{3^2+4^2} = 5$

$\cos\theta = \frac{-7}{5 \cdot 3\sqrt{3}} = \frac{-7\sqrt{3}}{45}$

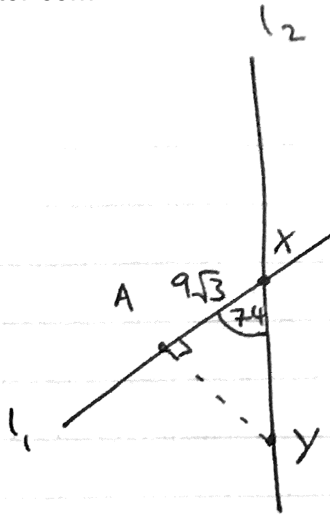
$\therefore \theta = \cos^{-1}\left(\frac{-7\sqrt{3}}{45}\right) = 105.63°$

so required angle $= 180 - 105.63 = \boxed{74.37°}$

c) $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 18 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ -15 \\ 3 \end{pmatrix}$

$\therefore |\overrightarrow{AX}| = \sqrt{3^2+15^2+3^2} = \boxed{9\sqrt{3}}$

d) $|\overrightarrow{AX}| = 9\sqrt{3}$



$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

$$\tan(74.37°) = \frac{AY}{9\sqrt{3}}$$

$$\therefore AY = 9\sqrt{3}\tan(74.37)$$

$$= \boxed{55.7}$$

e) $|\overrightarrow{AB}| = \frac{1}{2}|\overrightarrow{AX}|$

$$\therefore \overrightarrow{OB} = \overrightarrow{OA} \pm \frac{1}{2}\overrightarrow{AX}$$

$$\left.\begin{aligned}\overrightarrow{AX} &= \begin{pmatrix}-3\\-15\\3\end{pmatrix}\\ \overrightarrow{OA} &= \begin{pmatrix}2\\18\\6\end{pmatrix}\end{aligned}\right\} \quad \overrightarrow{OB_1} = \begin{pmatrix}2\\18\\6\end{pmatrix} + \begin{pmatrix}-3/2\\-15/2\\3/2\end{pmatrix} = \boxed{\begin{pmatrix}1/2\\21/2\\15/2\end{pmatrix}}$$

and $$\overrightarrow{OB_2} = \begin{pmatrix}2\\18\\6\end{pmatrix} - \begin{pmatrix}-3/2\\-15/2\\3/2\end{pmatrix} = \boxed{\begin{pmatrix}7/2\\51/2\\9/2\end{pmatrix}}$$

Q7a) $\frac{dh}{dt} = k(h-9)^{\frac{1}{2}}$

$h=130,\ \frac{dh}{dt} = -1.1$: $-1.1 = k(121)^{\frac{1}{2}}$

$\frac{-1.1}{11} = k = \boxed{-0.1}$

b) $\frac{dh}{dt} = -0.1(h-9)^{\frac{1}{2}}$

$\left(\frac{1}{(h-9)^{\frac{1}{2}}}\right)\frac{dh}{dt} = -0.1$

$\int\left[(h-9)^{-\frac{1}{2}}\right]dh = -0.1\int(1)\,dt$

$2(h-9)^{-1/2} = -0.1t + c$

$h=200,\ t=0$: $2(191)^{1/2} = c = 2\sqrt{191}$

$\therefore\ 2\sqrt{h-9} = 2\sqrt{191} - 0.1t$

$h=50$: $\frac{2\sqrt{41} - 2\sqrt{191}}{-0.1} = t = 148.343\ldots$

$= \boxed{148}$

Q8a) $y=8$: $8 = \sec^3\theta$

$\frac{1}{8} = \cos^3\theta \quad \therefore \cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

at $\theta = \frac{\pi}{3}$, $x = 3\left(\frac{\pi}{3}\right)\sin\frac{\pi}{3} = \boxed{\frac{\pi\sqrt{3}}{2}} = k$.

b) $R = \int\left(y\frac{dx}{d\theta}\right)d\theta \qquad \left[\frac{dx}{d\theta} = 3\theta\cos\theta + 3\sin\theta\right.$ (PRODUCT RULE) $\left.\right]$

$= \int_0^{\pi/3}\left[\sec^3\theta\cdot(3\theta\cos\theta + 3\sin\theta)\right]d\theta$

x	θ
0	0
$\frac{\pi\sqrt{3}}{2}$	$\frac{\pi}{3}$

$\leftarrow y = 8, \theta = \frac{\pi}{3}$

$= \int_0^{\pi/3}\left[3\theta\sec^2\theta + \frac{3\sin\theta}{\cos\theta}\cdot\sec^2\theta\right]d\theta$

$= 3\int_0^{\pi/3}\left[\theta\sec^2\theta + \tan\theta\sec^2\theta\right]d\theta$

$\lambda = 3$
$\alpha = 0$
$\beta = \frac{\pi}{3}$

c) $R = 3\int_0^{\pi/3} [\theta \sec^2\theta] \, d\theta + 3\int_0^{\pi/3} [\tan\theta \sec^2\theta] \, d\theta$

$\quad\quad\quad\quad\quad ① \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ②$

① By parts: $\frac{dv}{d\theta} = \sec^2\theta \longrightarrow v = \tan\theta$

$u = \theta \longrightarrow u' = 1$

$= 3\left[[\theta \tan\theta]_0^{\pi/3} - \int_0^{\pi/3} [\tan\theta] \, d\theta \right]$

$= 3\left[\frac{\pi}{3}\sqrt{3} - [\ln|\sec\theta|]_0^{\pi/3} \right]$

$= \pi\sqrt{3} - 3\ln 2$ //

② By pattern: $3\int_0^{\pi/3} [\tan\theta \sec\theta \cdot \sec\theta] \, d\theta$

$= 3\left[\frac{\sec^2\theta}{2} \right]_0^{\pi/3} = 3\left[2 - \frac{1}{2} \right] = \frac{9}{2}$ //

$R = ① + ② = \boxed{\pi\sqrt{3} - 3\ln 2 + \frac{9}{2}}$